Phys 312
Fall 2016

# Electromagnetic Theory II: Class Exam 2

9 November 2016

**Name:** ______________________________ **Total:** /50

## Instructions

- There are 4 questions on 6 pages.
- Show your reasoning and calculations and always explain your answers.

## Physical constants and useful formulae

| | |
|---|---|
| Permittivity of free space | $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$ |
| Permeability of free space | $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$ |
| Charge of an electron | $e = -1.60 \times 10^{-19}\,\mathrm{C}$ |
| Charge of a proton | $e = +1.60 \times 10^{-19}\,\mathrm{C}$ |
| Speed of light | $c = 3.0 \times 10^{8}\,\mathrm{m/s}$ |

## Question 1

Consider an electromagnetic wave in free space. The complex representation of a particular wave is

$$\widetilde{\mathbf{E}} = E_0 e^{i(ky-\omega t)}\hat{\mathbf{z}}$$

where $k, \omega$ and $E_0 > 0$ are real. **By substituting into one of Maxwell's equations,** find an expression for $\frac{\partial \widetilde{\mathbf{B}}}{\partial t}$. Solve this equation to determine an expression for $\widetilde{\mathbf{B}}$ in terms of $E_0, \omega, k$ and constants.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = \frac{\partial E_z}{\partial y}\hat{x} - \cancel{\frac{\partial E_z}{\partial x}}\hat{y}$$

$$= ik E_0 e^{i(ky-\omega t)}\hat{x} = -\frac{\partial \vec{B}}{\partial t}.$$

$$\frac{\partial \vec{B}}{\partial t} = -ik E_0 e^{i(ky-\omega t)}\hat{x}$$

$$\vec{B} = -ikE_0 \int e^{i(ky-\omega t)}\,dt\,\hat{x}$$

$$= \frac{-ik}{-i\omega} E_0 e^{i(ky-\omega t)}\hat{x}$$

$$\vec{B} = \frac{k}{\omega} E_0 e^{i(ky-\omega t)}\hat{x}$$

But $\omega = kc \Rightarrow \frac{k}{\omega} = \frac{1}{c} \Rightarrow \vec{B} = \frac{1}{c} E_0 e^{i(ky-\omega t)}\hat{x}$

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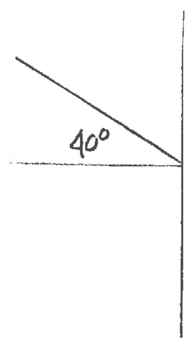
## Question 2

**Answer either part a) or part b) for full credit for this problem.**

a) Light in water (index of refraction 1.33) is incident on glass (index of refraction 1.50). at an angle of 40° with respect to the normal. Determine whether light polarized perpendicular to the plane of incidence or light polarized parallel to the plane of incidence is reflected more intensely.

$\theta_I = 40°$




Snells law $\Rightarrow \quad n_I \sin\theta_I = n_T \sin\theta_T$

$\Rightarrow \quad 1.33 \sin 40° = 1.50 \sin\theta_T$

$\Rightarrow \quad \theta_T = 35°$

Now for perpendicular polarization

$$R = \left(\frac{n_1 \cos\theta_I - n_2 \cos\theta_T}{n_1 \cos\theta_I + n_2 \cos\theta_T}\right)^2 = \left(\frac{1.02 \quad - 1.23}{1.02 \quad + 1.23}\right)^2 = \left(\frac{-0.21}{2.25}\right)^2$$

$$= 0.0087$$

For parallel polarization

$$R = \left(\frac{n_1 \cos\theta_T - n_2 \cos\theta_I}{n_1 \cos\theta_T + n_2 \cos\theta_I}\right)^2 = \left(\frac{1.09 - 1.15}{1.09 + 1.15}\right)^2 = \left(\frac{0.06}{2.24}\right)^2$$

$$= 0.0007$$

larger for <u>perpendicular</u>

Question 2 continued ...

b) An electromagnetic wave propagates in a medium with conductivity $\sigma$, permeability $\mu$ and permittivity $\epsilon$. The complex representation for the electric field is

$$\widetilde{\mathbf{E}}(z,t) = \widetilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{\mathbf{x}}$$

where $\widetilde{E}_0$ is independent of $x, y, z, t$,

$$k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2} \quad \text{and} \quad \kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

Suppose that $\omega \gg \sigma/\epsilon$. Determine an approximate (non-zero) expression for the skin depth. Does the depth to which the wave penetrates depend on the frequency?

$\frac{\sigma}{\omega\epsilon} \ll 1$

$d = \frac{1}{\kappa}$

Need $\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\left(1+\left(\frac{\sigma^2}{\epsilon\omega}\right)\right)^{1/2} - 1\right]^{1/2}$

$(1+x)^{1/2} \approx 1 + \frac{x}{2}$ for $x \ll 1$

This is the case here. So

$$\kappa \approx \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\cancel{1} + \left(\frac{\sigma^2}{\epsilon\omega}\right)\frac{1}{2} - \cancel{1}\right]^{1/2}$$

$$= \cancel{\omega}\sqrt{\frac{\epsilon\mu}{2}}\,\frac{\sigma}{\epsilon\cancel{\omega}}\,\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\,\sigma\sqrt{\frac{\mu}{\epsilon}}$$

$\Rightarrow \quad d \approx \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$

Not dependent on $\omega$.

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## Question 3

An arrangement of currents and charges produces the following potentials, given in cylindrical coordinates as

$$V = 0$$

and

$$\mathbf{A} = \begin{cases} 0 & \text{if } s < R \\ A_0 \cos(\omega t)\hat{\phi} & \text{if } s > R \end{cases}$$

where $A_0$ and $\omega$ are constants.

a) Determine the electric and magnetic fields produced by these potentials.

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$= 0 + \omega A_0 \sin(\omega t)\,\hat{\phi} \quad \Rightarrow \quad \vec{E} = +\omega A_0 \sin(\omega t)\,\hat{\phi}$$

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$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{A} = A_\phi \hat{\phi} \qquad A_\phi = A_0 \cos\omega t$$

$$= \cancel{-\frac{\partial A_\phi}{\partial z}\hat{s}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s A_\phi)\right]\hat{z}$$

$$\Rightarrow \quad \vec{B} = \frac{1}{s} A_0 \cos(\omega t)\,\hat{z}$$

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b) In which direction do these fields transport energy? Explain your answer.

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

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$$= \frac{1}{\mu_0}(+\omega A_0 \sin\omega t)\left(\frac{1}{s}A_0 \cos\omega t\right)\hat{\phi} \times \hat{z}$$

$$= +\frac{\omega A_0^2}{\mu_0 s} \sin\omega t \cos\omega t\,\hat{s}$$

In and out along $\hat{s}$

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## Question 4

**Answer either part a) or part b) for full credit for this problem.**

a) Consider the potentials given Cartesian coordinates as

$$V = 0$$

and

$$\mathbf{A} = A_0 \cos(kz - \omega t)\hat{\mathbf{z}} \quad \hat{y}$$

where $A_0$ and $\omega$ are constants. Describe whether this potential is in the Coulomb gauge, the Lorentz gauge or both. Explain your answer.

Coulomb $\vec{\nabla}\cdot\vec{A} = 0\,?$ $\quad \vec{\nabla}\cdot\vec{A} = 0$

Is in Coulomb gauge.

Lorentz $\underbrace{\vec{\nabla}\cdot\vec{A}}_{=0} = -\mu_0\epsilon_0 \underbrace{\frac{\partial V}{\partial t}}_{=0}$

Is also in Lorentz gauge.

b) A stationary point particle is at the origin. At time $t_1$ it becomes positively charged and its charge stays constant until at a later time $t_2$ it becomes neutral again. An observer, who is stationary at a distance $R$ from the origin can measure electric and magnetic fields at her location. At what times does she detect electric fields? At what times does she detect magnetic fields? Explain your answers.

Using retarded potentials $V(\vec{r},t)$, $\vec{A}(\vec{r},t)$ are only non zero when a signal traveling from source at speed of light reaches. These are times

$t_1 + R/c$ to $t_2 + R/c$

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In these times $\vec{E} \neq 0$. The current is radially in/out giving $\vec{A} = A_r\hat{r}$ $\Rightarrow \vec{B} = 0$ always.