Phys 312
Fall 2016

Electromagnetic Theory II: Class Exam 1

23 September 2016

Name: SOLUTION **Total:** **/50**

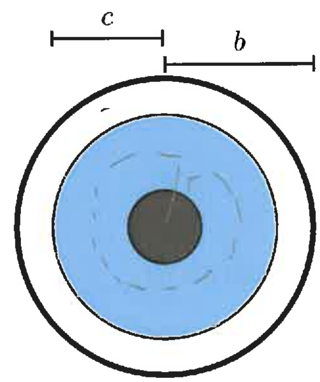
Instructions

- There are 3 questions on 6 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Charge of a proton	$e = +1.60 \times 10^{-19}\,\mathrm{C}$

Question 1



Two spherical conductors are arranged as illustrated (viewed along their axis). The radius of the inner conductor is a and of the outer conductor is b. A spherical dielectric material with radius c and permittivity $1.5\epsilon_0$ partly fills the region between the conductors. The inner ~~cylinder~~ carries a total charge Q and the outer ~~cylinder~~ sphere carries total charge $-Q$.

a) Determine the electric field at all locations beyond beyond the inner conductor.

The electric field and polarization will be spherically symmetrical. So

$$\vec{D} = D_r(r)\,\hat{r} \quad]+1$$

Then

$$\oint \vec{D}\cdot d\vec{a} = Q_{\text{free enc}} \quad]+1$$

Choose as a surface a sphere of radius $r > a$. Then on this

$0 \le \phi \le 2\pi$ $\quad$ $d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$0 \le \theta \le \pi$

and $\vec{D}\cdot d\vec{a} = D_r(r)\, r^2 \sin\theta \, d\theta \, d\phi$. So

$$\oint \vec{D}\cdot d\vec{a} = D_r(r)\, r^2 \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} d\theta \sin\theta}_{2} = 4\pi r^2 D_r(r)$$

+3

Now if $a < r < b$ $\quad$ $Q_{\text{free enc}} = Q$

if $b < r$ $\quad$ $Q_{\text{free enc}} = 0$

+2

$$\Rightarrow \quad \vec{D} = \begin{cases} \dfrac{Q}{4\pi r^2}\,\hat{r} & a < r < b \quad]+1 \\ 0 & b < r \quad]+1 \end{cases}$$

Question 1 continued . . .

Now in any region $\vec{E} = \vec{D}/\epsilon_{region}.$ +1

Thus for $a < r < c$ $\quad \epsilon_{region} = 1.5\epsilon_0$

$$\Rightarrow \quad \vec{E} = \frac{1}{6\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \qquad +1$$

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for $c < r < b$ $\quad \epsilon_{region} = \epsilon_0$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \qquad +1$$

for $b < r$

$$\vec{E} = 0 \qquad +1$$

b) Determine the bound surface and volume charge densities throughout the dielectric material in terms of Q.

+1 $\sigma_b = \vec{P}\cdot\hat{n}$

+1 $\rho_b = -\vec{\nabla}\cdot\vec{P}$

Then $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$$\Rightarrow \quad \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P}$$

$$\Rightarrow \quad \vec{P} = (\epsilon - \epsilon_0)\vec{E}$$

$$\vec{P} = 0.5\epsilon_0\vec{E} \qquad +2$$

+2 So on inner surface $\hat{n} = -\hat{r}$ $\quad \sigma_b = -0.5\epsilon_0\vec{E}\cdot\hat{r}$ at $r = a$

$$\sigma_b = -\frac{1}{12\pi}\frac{Q}{a^2}$$

+2 on outer surface $\hat{n} = \hat{r}$, $r = c$ $\quad \sigma_b = 0.5\epsilon_0\vec{E}\cdot\hat{r}$ at $r = c$

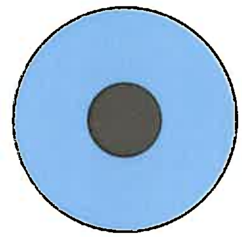
$$\sigma_b = \frac{1}{12\pi}\frac{Q}{c^2}$$

+1 in volume $\quad \vec{\nabla}\cdot\vec{P} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 0.5\epsilon_0 E_r)$

$$= \frac{0.5\epsilon_0}{r^2}\frac{\partial}{\partial r}\left(\frac{\cancel{r^2} Q}{6\pi\epsilon_0\cancel{r^2}}\right) = 0 \qquad /22$$

$$\Rightarrow \quad \rho_b = 0$$

Question 2



An infinitely long cylinder with radius a carries a current I along the length of the cylinder. The cylinder is surrounded by a cylindrical linear material with radius b. The arrangement is illustrated as viewed down the axis of the cylinder with the current in the cylinder pointing out of the page.

a) Suppose that the permeability of the material is $\mu > \mu_0$. Describe the direction of the bound surface currents on the inner surface of the material surrounding the rod. Explain your answer.

Need $\vec{K}_f = \vec{M} \times \hat{n}$ +1 with $\hat{n} = -\hat{s}$ +1. This requires $\vec{M}$.

But $\vec{B} = \mu \vec{H}$ and $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

$$\Rightarrow \quad \vec{H} = \frac{\mu}{\mu_0}\vec{H} - \vec{M} \quad \Rightarrow \quad \vec{M} = \left(\frac{\mu}{\mu_0} - 1\right)\vec{H}$$

+2

In this case by Biot-Savart, current reversal and symmetry +2

$\vec{H} = H_\phi(s)\hat{\phi}$ with $H_\phi > 0$

$$\Rightarrow \quad \vec{K}_f = \left(\frac{\mu}{\mu_0} - 1\right) H_\phi \underbrace{\hat{\phi} \times (-\hat{s})}_{\hat{z}}$$

$$= \left(\frac{\mu}{\mu_0} - 1\right) H_\phi \hat{z}$$

+2

If $\mu > \mu_0$ this is in $+\hat{z}$ direction +1

b) Suppose that the permeability of the material is $\mu < \mu_0$. Describe the direction of the bound surface currents on the inner surface of the material surrounding the rod. Explain your answer.

The derivation above holds by $\mu < \mu_0 \Rightarrow \vec{K}_f$ is in $-\hat{z}$ direction. +3

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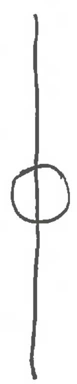
Question 3

An infinitely long straight wire along the z axis carries a uniform current with magnitude I (this is such that the charge density along the wire is zero). A spherical shell with radius R is centered at the origin and carries a stationary charge Q that is uniformly distributed along its surface.

a) Determine the electric and magnetic fields produced by these charges and currents at all locations. Determine the Poynting vector at all locations. Describe the direction in which electromagnetic energy flows.



Sphere of charge: Symmetry $\Rightarrow$ $\vec{E} = E_r \hat{r}$ (+1)

$$\oint \vec{E}\cdot d\vec{a} = Q_{enc}/\epsilon_0 \quad (+1)$$

Use a sphere of radius r as a Gaussian surface

If $r<R$ $\quad Q_{enc}=0$

$r>R$ $\quad Q_{enc}=Q$

$$\oint \vec{E}\cdot d\vec{a} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta\, E_r$$

$$= 4\pi r^2 E_r \quad (+3)$$

Thus

$$\vec{E} = \begin{cases} 0 & r<R \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} & r>R \end{cases} \quad (+2)$$

The magnetic field satisfies $\vec{B} = B_\phi(s)\,\hat{\phi}$ (+1)

Then

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow 2\pi s B_\phi = \mu_0 I$$

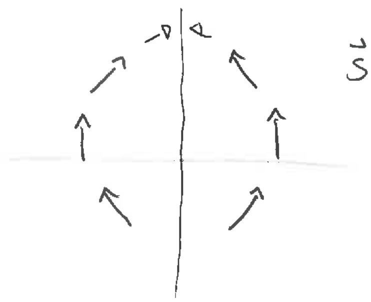
circular loop radius s ... perpendicular to current

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad (+2)$$

Question 3 continued ...

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \begin{cases} 0 & \text{inside} \\ \dfrac{IQ}{8\pi^2\epsilon_0 s r}\,\underbrace{\hat{r}\times\hat{\phi}}_{-\hat{\theta}} & \end{cases}$$ +2

Points along $-\hat{\theta}$



+1

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b) Describe the sign of the work done by the electric field on the current (at various locations). Verify that this is consistent with the direction of energy flow.

$W = \int \vec{E}\cdot\vec{J}\,dz$ +1

For $z > 0$ $\quad \vec{J}\uparrow \uparrow\vec{E} \Rightarrow W > 0$

$z < 0$ $\quad \vec{J}\uparrow\downarrow\vec{E} \Rightarrow W < 0$

+2

energy is removed from $z<0$

added to $z>0$

consistent with energy flow from Poynting vector $\vec{S}$

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