Fri: HW 5pm

Tues: Read ~~9.2.3~~ 9.2.2, 9.2.3, 9.3.1, 9.3.2

Classical waves in three dimensions

Waves can propagate in two and three dimensions. In the simplest case the disturbance is a scalar. For example:

1) pressure at any location and time $P(x,y,z,t)$

Denote the generic displacement by

$$f(\vec{r},t) = f(x,y,z,t)$$

Then the three-dimensional wave equation is:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

and this is written:

$$\boxed{\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

We will consider solutions to the scalar three dimensional wave equation. Note that positions are represented by

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

1 Three dimensional wave equation

Consider the three dimensional wave equation

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$

a) Show that

$$f(\mathbf{r},t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

where $A > 0$ is a solution given that $\mathbf{k}$ and ω satisfy a particular condition. Describe the condition.

b) Suppose that $\mathbf{k} = k\hat{\mathbf{x}}$. Determine surfaces along which $f(\mathbf{r},t)$ attains a maximum. Describe the shape of these surfaces, the direction in which they propagate and the speed with which the propagate.

c) Suppose that $\mathbf{k} = k(\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. Determine surfaces along which $f(\mathbf{r},t)$ attains a maximum. Describe the shape of these surfaces, the direction in which they propagate and the speed with which the propagate.

Answer: a) $f(\vec{r},t) = A\cos\left(k_x x + k_y y + k_z z - \omega t\right)$

$$\frac{\partial^2 f}{\partial x^2} = -k_x^2 f(\vec{r},t) \qquad \frac{\partial^2 f}{\partial t^2} = -\omega^2 f(\vec{r},t)$$

$$\frac{\partial^2 f}{\partial y^2} = -k_y^2 f(\vec{r},t)$$

$$\frac{\partial^2 f}{\partial z^2} = -k_z^2 f(\vec{r},t)$$

Substitution gives

$$-(k_x^2 + k_y^2 + k_z^2) = -\omega^2/v^2 \quad \Rightarrow \quad \vec{k}\cdot\vec{k} = \omega^2/v^2$$

or $\omega = v\sqrt{\vec{k}\cdot\vec{k}}$

b) Maximum is attained when $\vec{k}\cdot\vec{r} - \omega t = 2n\pi$ where n is an integer.

Here $\vec{k}\cdot\vec{r} = kx$

$$\Rightarrow \quad kx - \omega t = 2n\pi \quad \Rightarrow \quad x = \frac{\omega}{k}t + 2n\pi/k.$$

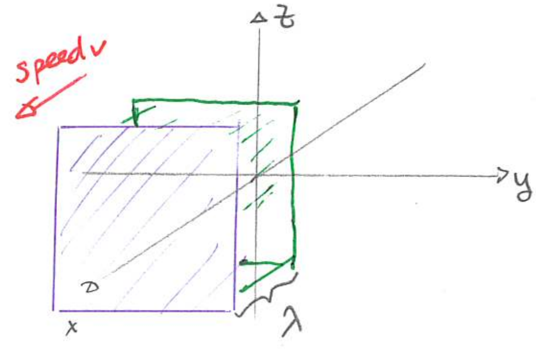
Thus maxima occur when

$$x = vt + 2n\pi/k$$

These are planes along which x = constant $\Rightarrow$ planes parallel to yz.

At any instant adjacent planes are separated by distance

$$\Delta x = 2\pi/k \Rightarrow \Delta x = \lambda$$



The planes travel with speed v

c) Maxima occur at $\vec{k}\cdot\vec{r} - \omega t = 2n\pi$

$$\Rightarrow k_x x + k_y y = \omega t + 2n\pi \qquad n = \text{integer.}$$

$$\Rightarrow \frac{k}{\sqrt{2}}(x+y) = \omega t + 2n\pi$$

$$\Rightarrow (x+y)/\sqrt{2} = vt + \frac{2n\pi}{k}.$$

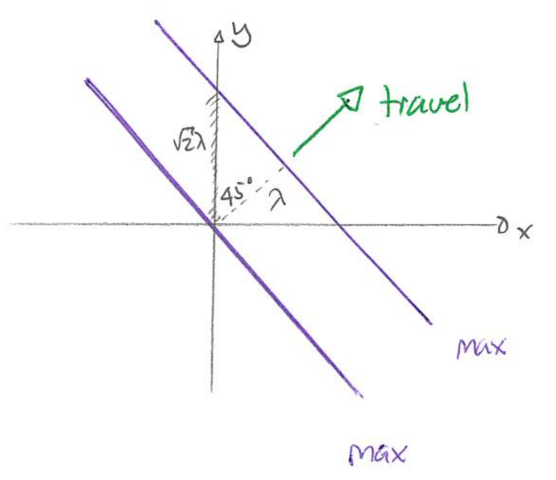
$$\Rightarrow x+y = \sqrt{2}vt + \sqrt{2}\frac{2n\pi}{k}.$$

$$\Rightarrow y = -x + \sqrt{2}vt + \sqrt{2}n\lambda$$

At any instant these are planes at a 45° angle between x, y axes. For example at $t=0$ $y = -x + \sqrt{2}n\lambda$

As t increases the slope stays constant but the y-intercept increases. This means the planes propagate along the direction

$$\frac{1}{\sqrt{2}}(\hat{x}+\hat{y}) = \hat{k}$$



The exercise illustrates two examples of plane waves, solutions where the surfaces of constant f are planes.

The general plane sinusoidal wave solution to the three dimensional wave equation is

$$f(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

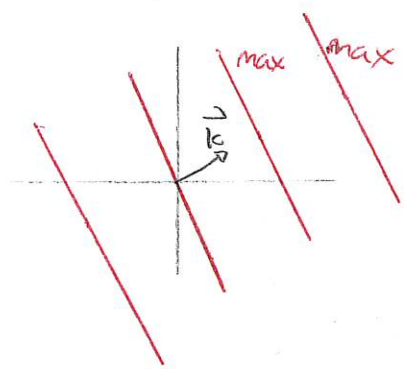
where A, ω, δ are constants (usually positive) and $\vec{k}$ is a vector. This has properties

1) $\omega = kv = \sqrt{\vec{k}\cdot\vec{k}}\, v$

2) the surfaces along which f is constant are planes perpendicular to $\vec{k}$

3) the distance between successive maximal planes is $\lambda = 2\pi/k$

4) the disturbance propagates along $\vec{k}$ with speed $v = \omega/k$.



The complex representation of these is

$$\boxed{\tilde{f}(\vec{r},t) = \tilde{A}e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \text{where} \quad \tilde{A} = Ae^{i\delta}}$$

The generic plane wave solution to the three dimensional wave equation is

$$f(\vec{r},t) = g(\vec{k}\cdot\vec{r} - \omega t) + h(\vec{k}\cdot\vec{r} + \omega t)$$

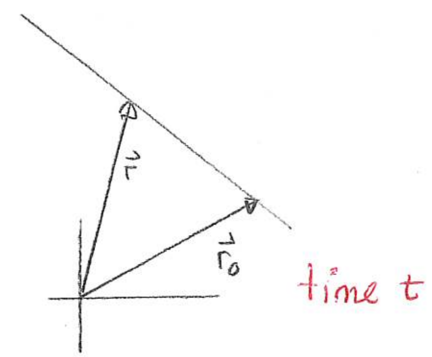
where g, h are arbitrary functions of a single variable

We can prove this as follows. Consider $g(\vec{k}\cdot\vec{r}-\omega t)$. This returns a single constant value whenever

$$\vec{k}\cdot\vec{r}-\omega t = \alpha = \text{constant}.$$

$$\Rightarrow \vec{k}\cdot\vec{r} = \omega t + \alpha$$

This is the equation of a plane perpendicular to $\vec{k}$. The reason is that. Let $\vec{r}_0$ be any vector to this plane. Then let $\vec{r}$ be a vector to an arbitrary point. So



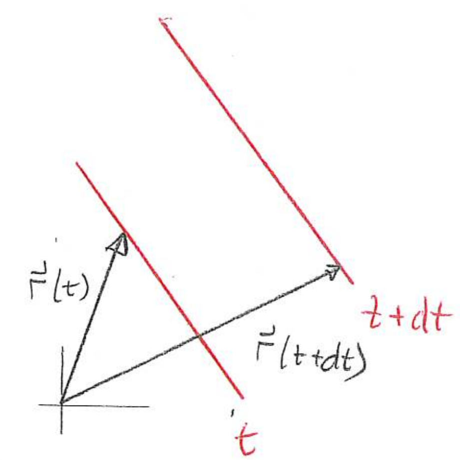
$$(\vec{r}-\vec{r}_0)\cdot\vec{k} = 0$$

$$\Rightarrow \vec{r}\cdot\vec{k} = \underbrace{\vec{r}_0\cdot\vec{k}}_{\text{constant}}$$

$$\Rightarrow xk_x + yk_y + zk_z = \underbrace{\vec{r}_0\cdot\vec{k}}_{\text{constant}}$$

co-ords of any pt on plane.

Now consider how the plane propagates from time $t \to t+dt$. Let $\vec{r}(t)$ be a vector to a point on the plane at time t. Let $\vec{r}(t+dt)$ be a vector to a point on the plane at time $t+dt$. So



$$\vec{r}(t+dt)\cdot\vec{k} = \omega(t+dt)+\alpha$$

$$\vec{r}(t)\cdot\vec{k} = \omega t + \alpha$$

$$\Rightarrow \left[\vec{r}(t+dt) - \vec{r}(t)\right]\cdot\vec{k} = \omega\, dt$$

Let $\Delta\vec{r} = \vec{r}(t+dt) - \vec{r}(t)$. $\Rightarrow (\Delta\vec{r})\cdot\vec{k} = \omega\, dt$

The distance between the planes is the smallest $\Delta \vec{r}$ that satisfies this. Then let θ be the angle between $\Delta \vec{r}$ and $\vec{k}$

$$\Delta r\, k \cos\theta = \omega\, dt \Rightarrow \Delta r = \frac{\omega\, dt}{k \cos\theta}$$

Δr is minimized when $\cos\theta$ is maximized. This occurs when $\cos\theta = 1$

Thus $\theta = 0$. So the wave travels parallel to $\vec{k}$. The speed is

$$\lim \frac{\Delta r}{dt} = \omega/k = v$$

The wave travels with velocity

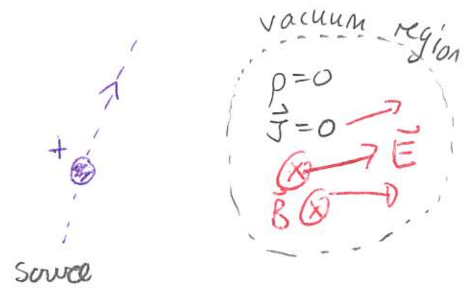
$$\vec{v} = v\,\hat{k} = \omega/k\;\hat{k}$$

where $\hat{k}$ is a unit vector along $\vec{k}$.

Electromagnetic wave equations

Electric and magnetic fields can exist in a region with no source charges or currents. The sources could be elsewhere.

So we consider Maxwell's equations in such regions. Thus



$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

become:

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

These are first order differential equations. The $\vec{E}$ and $\vec{B}$ fields are coupled. One strategy to uncouple these is to differentiate again. This will require the identity:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

Then consider

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

$$\Rightarrow \cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})} - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

$$\Rightarrow \nabla^2\vec{E} = \frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

$$\Rightarrow \quad \nabla^2 \vec{E} = \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \quad \nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

A similar derivation gives

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Thus in a region where there are no source currents or charges:

$$\boxed{\begin{aligned} \nabla^2 \vec{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla^2 \vec{B} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \end{aligned}}$$

These are actually each a set of three wave equations, one for each field component:

$$\nabla^2 E_x = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \qquad \qquad \nabla^2 B_x = \mu_0 \epsilon_0 \frac{\partial^2 B_x}{\partial t^2}$$

$$\nabla^2 E_y = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} \qquad \qquad \vdots$$

$$\vdots$$

They all have the form

$$\nabla^2 f = \frac{1}{V^2} \frac{\partial^2 f}{\partial t^2}$$

where the wave speed is

$$V = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Substituting known values for these constants gives a speed close to the measured speed of light.

Thus:

> In a vacuum region (no source currents or charges), Maxwell's equations imply that waves of electric and magnetic fields exist. These travel with the speed of light
>
> $$c = 1/\sqrt{\mu_0 \epsilon_0}$$

These are electromagnetic waves

DEMO: 1) PSU-S
2) PhET radio waves.

We will establish general rules that such electromagnetic waves must satisfy. We will illustrate this with sinusoidal waves.

Sinusoidal electromagnetic traveling waves

Sinusoidal waves will have the form:

$$\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$$

where $\vec{E}_0$ is a vector independent of $\vec{r}$ and t. Here $\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$ is a wavenumber vector and ω is an angular frequency. The complex exponential version is:

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \xrightarrow{\text{real solutions}} \vec{E} = Re[\tilde{\vec{E}}]$$

$$\tilde{\vec{B}} = \tilde{\vec{B}}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \longrightarrow \vec{B} = Re[\tilde{\vec{B}}].$$

We ask:

1) are the solutions for electric and magnetic fields independent?
2) how are the directions of the fields related to the direction of propagation?
3) how are the directions of $\vec{E}$ and $\vec{B}$ related?

2 Electromagnetic sinusoidal plane waves

Consider

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$$

where $\tilde{\mathbf{E}}_0$ is constant.

a) Using the divergence of the electric field, determine constraints on the direction of $\tilde{\mathbf{E}}_0$.

b) Using the curl of the electric field, show that the magnetic field wave propagates in the same direction as the electric field and with the same wavenumber and frequency.

c) If $\tilde{\mathbf{E}}_0 = \tilde{E}_0\hat{\mathbf{x}}$ use the curl of the electric field to determine the direction and magnitude of the magnetic field.

Answer: a) $\vec{\nabla}\cdot\tilde{\vec{E}} = 0$

$$\Rightarrow \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y} + \frac{\partial \tilde{E}_z}{\partial z} = 0$$

and $\tilde{E}_x = \tilde{E}_{0x} e^{i(kz-\omega t)} \Rightarrow \frac{\partial \tilde{E}_x}{\partial x} = 0$

$\tilde{E}_y = \tilde{E}_{0y} e^{i(kz-\omega t)} \Rightarrow \frac{\partial \tilde{E}_y}{\partial y} = 0$

$\tilde{E}_z = \tilde{E}_{0z} e^{i(kz-\omega t)} \Rightarrow \frac{\partial \tilde{E}_z}{\partial z} = \tilde{E}_{0z} e^{i(kz-\omega t)}\, ik = ik\tilde{E}_z$

Thus $\vec{\nabla}\cdot\tilde{\vec{E}} = ik\tilde{E}_z = 0 \Rightarrow \tilde{E}_z = 0$

$\Rightarrow \tilde{\vec{E}}$ is in the xy plane

$\Rightarrow \tilde{\vec{E}}$ is perpendicular to z

$\Rightarrow \tilde{\vec{E}}$ is perpendicular to direction of propagation.

b) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E}_x & \tilde{E}_y & \tilde{E}_z \end{vmatrix} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \quad \hat{x}\left(-\frac{\partial \tilde{E}_y}{\partial z}\right) + \hat{y}\,\frac{\partial \tilde{E}_x}{\partial z} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \quad -ik\,\tilde{E}_{oy}\,e^{i(kz-\omega t)}\,\hat{x} + ik\,\tilde{E}_{ox}\,e^{i(kz-\omega t)}\,\hat{y} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \vec{B}}{\partial t} = ik\,\tilde{E}_{oy}\,e^{i(kz-\omega t)}\,\hat{x} - ik\,\tilde{E}_{ox}\,e^{i(kz-\omega t)}\,\hat{y}$$

$$\Rightarrow \begin{cases} \dfrac{\partial B_x}{\partial t} = ik\tilde{E}_{oy}\,e^{i(kz-\omega t)} \\ \dfrac{\partial B_y}{\partial t} = -ik\,\tilde{E}_{ox}\,e^{i(kz-\omega t)} \\ \dfrac{\partial B_z}{\partial t} = 0 \end{cases} \Rightarrow \begin{cases} B_x = \dfrac{ik}{-i\omega}\,\tilde{E}_{oy}\,e^{i(kz-\omega t)} \\ B_y = \dfrac{-ik}{-i\omega}\,\tilde{E}_{ox}\,e^{i(kz-\omega t)} \\ B_z = 0 \end{cases}$$

Then $k/\omega = \frac{1}{c}$ gives:

$$\tilde{\vec{B}} = -\frac{1}{c}\left[\tilde{E}_{oy}\,e^{i(kz-\omega t)}\hat{x} - \tilde{E}_{ox}\,e^{i(kz-\omega t)}\hat{y}\right]$$

This is a wave propagating along the z axis with wavenumber k and frequency ω. The wave is also perpendicular to z. Note that an overall integration constant $\tilde{\vec{B}}_1$ independent of t was omitted.

Then $\vec{\nabla}\cdot\tilde{\vec{B}} = 0 \Rightarrow \vec{\nabla}\cdot\tilde{\vec{B}}_1 = 0$
$\vec{\nabla}\times\tilde{\vec{B}} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times\vec{B}_1 = 0$
} Helmholtz theorem $\Rightarrow \vec{B}_1 = 0$

c) $\tilde{\vec{B}} = \frac{1}{c}\tilde{E}_0\,e^{i(kz-\omega t)}\,\hat{y}$ magnitude $\frac{E_0}{c}$

direction along $\vec{y}$