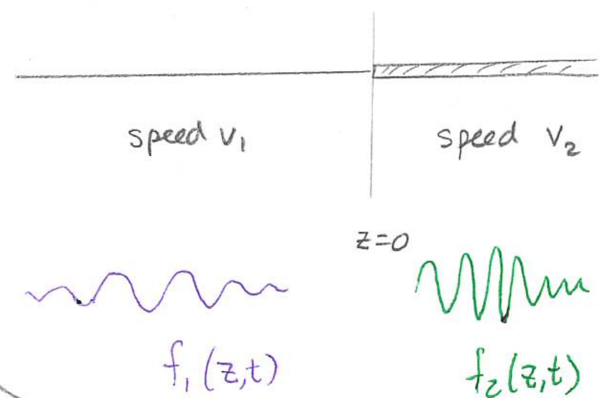
Tues: HW by 5pm Thurs 9.2.1 → 9.2.2

Fri: HW by 5pm

## Reflection and transmission of waves in one dimension

We consider waves on a one dimensional medium such as a string which has a discontinuity in speed at $z=0$. Waves can be reflected and transmitted at the boundary. In general we have that



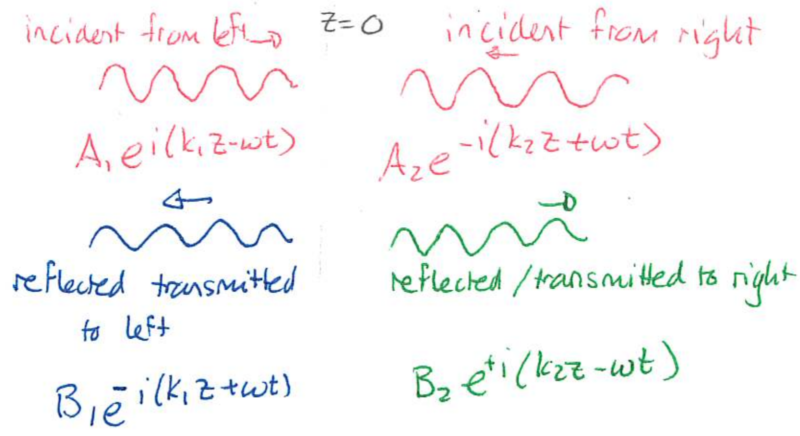

Let $f_1(z,t)$ be a solution when $z<0$ and $f_2(z,t)$ be a solution when $z>0$. Then at the boundary

$$f_1(0,t) = f_2(0,t) \quad \text{AND} \quad \left.\frac{\partial f_1}{\partial z}\right|_{z=0} = \left.\frac{\partial f_2}{\partial z}\right|_{z=0}$$

We can interpret the transmission and reflection by considering sinusoidal waves of one frequency $\omega$. Then the wavenumbers are

$$k_1 = \omega/v_1 \qquad z<0$$

$$k_2 = \omega/v_2 \qquad z>0$$



After applying matching conditions this gives

To the left

$$f_1(z,t) = A_1 e^{i(k_1 z - \omega t)} + B_1 e^{-i(k_1 z + \omega t)}$$

to the right

$$f_2(z,t) = B_2 e^{i(k_2 z - \omega t)} + A_2 e^{-i(k_2 z + \omega t)}$$

where

$$B_1 = \frac{k_1 - k_2}{k_1 + k_2} A_1 + \frac{2k_2}{k_1+k_2} A_2$$

$$B_2 = \frac{2k_1}{k_1+k_2} A_1 + \frac{k_2 - k_1}{k_1 + k_2} A_2$$

## Waves incident from one side

In general we consider waves incident from one side. Suppose that waves are incident from the left. Then $A_2 = 0$. Thus

$$B_1 = \frac{k_1 - k_2}{k_1 + k_2} A_1 \quad \rightsquigarrow \quad \text{amplitude of reflected wave}$$

$$B_2 = \frac{2k_1}{k_1+k_2} A_1 \quad \rightsquigarrow \quad \text{" \quad " transmitted wave}$$

We first explore these in extreme cases.

## 1 Reflection and transmission along a string: extreme cases

Consider sinusoidal waves with frequency $\omega$ that are incident on a junction at $z = 0$. The wave speed for $z < 0$ is $v_1$ and the wave speed for $z > 0$ is $v_2$. The wavenumbers are $k_i = \omega/v_i$. Suppose that waves are incident from the left.

a) Suppose that the wave speeds on either side of the barrier are the same. What would you predict for the amplitudes of the reflected and transmitted waves in comparison to the incident wave?

b) Check whether the formulas agree with your predictions.

c) Suppose that $v_2 \ll v_1$. What would you predict for the amplitudes of the reflected and transmitted waves in comparison to the incident wave?

d) Check whether the formulas agree with your predictions.

a) The string has no discontinuity. We would expect no interruption of the incident wave $\Rightarrow$ NO reflection, perfect transmission.

b) $k_1 = k_2 \Rightarrow B_1 = 0$ no reflection ✓

$B_2 = A_1$ perfect transmission ✓

c) We would expect the wave to travel far more easily in the $z<0$ region than $z>0$. $\Rightarrow$ ALL reflected, NONE transmitted.

d) $k_2 \gg k_1$. Thus

$B_1 \approx \frac{-k_2}{k_2} A_1 \Rightarrow B_1 = -A_1$ pefect reflection with inversion

$B_2 \approx \frac{2k_1}{k_2} A_1 \approx 0 \Rightarrow B_2 \approx 0$ no transmission

DEMO: PhET W.o.a.S with fixed end.

## 2 Reflection and transmission of waves along a string

Consider waves on a string with a discontinuity at $z = 0$. Suppose that the waves are incident from the left with amplitude $A > 0$.

a) Determine conditions under which the reflected waves have "negative" amplitude. In such circumstances, what would the negative amplitude imply?

b) Does the transmitted wave ever have a "negative" amplitude?

c) Determine conditions under which the amplitude of the transmitted wave is greater than that of the incident wave.

d) Can the reflected wave ever have a greater amplitude than the transmitted wave?

a) We need $B_1 < 0 \Rightarrow k_1 < k_2 \Rightarrow \frac{\omega}{v_1} < \frac{\omega}{v_2}$

$\Rightarrow v_2 < v_1$

Passing from faster to slower medium $\Rightarrow$ waves are inverted

b) No $k_1 > 0 \Rightarrow B_1 > 0$

c) $\frac{2k_1}{k_1+k_2} \geq 1 \Rightarrow 2k_1 \geq k_1 + k_2 \Rightarrow k_1 \geq k_2 \Rightarrow v_2 \geq v_1$

If the wave travels into a medium with higher speed then the transmitted wave has a larger amplitude

d) Needs $\frac{2k_1}{k_1+k_2} < \left|\frac{k_1-k_2}{k_1+k_2}\right| \Leftrightarrow 2k_1 < |k_1-k_2|$ if $k_1 > k_2$ then

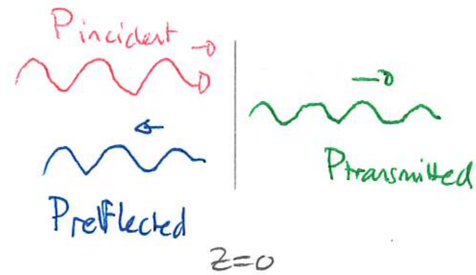
$2k_1 < k_1 - k_2 \Rightarrow k_2 < -k_1$ impossible

if $k_1 < k_2 \Rightarrow 2k_1 < k_2 - k_1 \Rightarrow 3k_1 < k_2$, it is possible

$\Rightarrow 3\omega/v_1 < \omega/v_2 \Rightarrow v_2 < v_1/3$

DEMO: Physclips UNSW Waves 2 $\rightarrow$ Reflection + transmission ... video

It appears that the transmitted wave can have a larger amplitude than the incident wave and that this might violate energy conservation. But we need to consider the rate at which energy enters and leaves $z=0$.




Thus we need to compare the reflected power to the incident power and similarly for the transmitted power.

In general the power for a sinusoidal wave is

$$P = \frac{T\omega^2}{v} A^2 \sin^2(kz-\omega t)$$

$A$ ↳ amplitude of wave.

Then

$$P_{incident} = \frac{T\omega^2}{v_1} A_1^2 \sin^2(kz-\omega t) = \frac{T\omega^2}{v_1} A_1^2 \sin^2(\omega t)$$

$$P_{reflected} = \frac{T\omega^2}{v_1} B_1^2 \sin^2(kz+\omega t) = \frac{T\omega^2}{v_1} B_1^2 \sin^2(\omega t)$$

$$P_{transmitted} = \frac{T\omega^2}{v_2} B_2^2 \sin^2(kz-\omega t) = \frac{T\omega^2}{v_2} B_2^2 \sin^2(\omega t)$$

at $z=0$.

→ note $v_2$

Thus

$$\frac{P_{reflected}}{P_{incident}} = \frac{B_1^2}{A_1^2} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2} \leq 1 \quad \text{always}$$

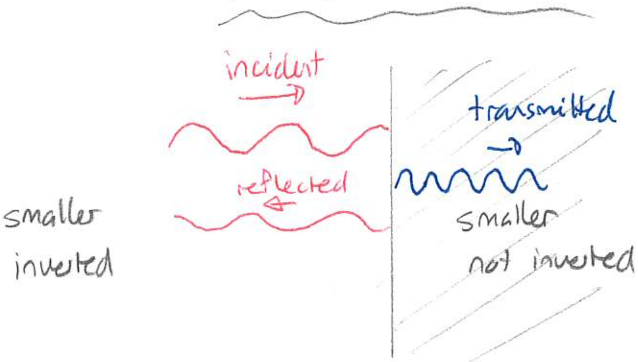
$$\frac{P_{transmitted}}{P_{incident}} = \frac{B_2^2}{A_1^2}\frac{v_1}{v_2} = \frac{(2k_1)^2}{(k_1+k_2)^2}\frac{\omega/k_1}{\omega/k_2} = \frac{4k_1k_2}{(k_1+k_2)^2}$$

we can show with algebra that the latter is less than 1. Thus

$$\frac{P_{transmitted}}{P_{incident}} \leq 1$$

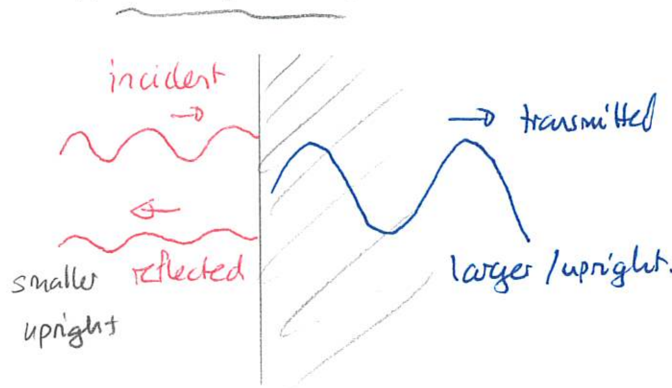
To summarize:

If $v_2 < v_1$




$k_2 > k_1 \Rightarrow \lambda_2 < \lambda_1$

If $v_2 < v_1$

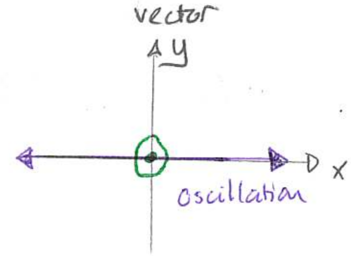



$k_2 < k_1 \Rightarrow \lambda_2 > \lambda_1$

# Polarization

The waves that we have considered all involve an oscillating scalar quantity $f(z,t)$. However, it is possible to have oscillating vector quantities such as:

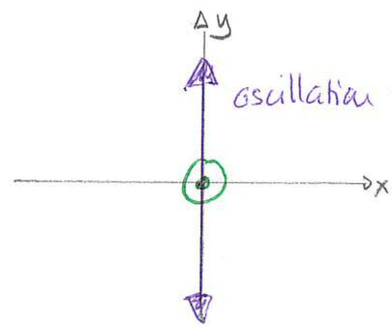
$$\vec{f}(z,t) = \underbrace{A\hat{x}}_{\text{vector}} \cos(kz-\omega t)$$




wave propagates outward

disturbance oscillates horizontally

$$\vec{f}(z,t) = A\hat{y} \cos(kz-\omega t)$$




wave propagates outward along $+z$

disturbance oscillates vertically

We say that such waves are polarized. Then

$$\boxed{\text{Polarization} \Longrightarrow \text{information about direction of oscillation}}$$

These are examples of transverse waves (oscillation is perpendicular to propagation) which are linearly polarized. In general a linearly polarized wave has the form:

$$\vec{f}(z,t) = A\hat{n} \cos(kz-\omega t)$$

where $\hat{n}$ is a unit vector indicating the direction of oscillation (or the axis of polarization). The complex representation of such waves is

$$\boxed{\tilde{\vec{f}}(z,t) = \tilde{A}\,\hat{n}\, e^{i(kz-\omega t)}}$$

where $\tilde{A}$ is complex

Then the most general transverse wave that travels along $+z$ is:

$$\boxed{\begin{aligned} &\tilde{\vec{f}}(z,t) = \tilde{\vec{A}}\, e^{i(kz-\omega t)} \\ &\text{where} \quad \tilde{\vec{A}} = \tilde{A}_x \hat{x} + \hat{A}_y \hat{y} \\ &\text{and} \quad \tilde{A}_x = A_x e^{i\delta_x}, \\ &\qquad\quad \tilde{A}_y = A_y e^{i\delta_y} \end{aligned}}$$

Thus

$$\tilde{\vec{f}}(z,t) = A_x e^{i(kz-\omega t+\delta_x)}\hat{x} + A_y e^{i(kz-\omega t+\delta_y)}\hat{y}$$

and the real wave is:

$$\vec{f}(z,t) = A_x \cos(kz-\omega t+\delta_x)\hat{x} + A_y \cos(kz-\omega t+\delta_y)\hat{y}$$

We analyze this by considering oscillations at one location as time passes.

## 3 Polarization of waves

The complex representation of "vector" sinusoidal waves is:

$$\tilde{\mathbf{f}} = \tilde{\mathbf{A}} e^{i(kz-\omega t)}$$

where $\tilde{\mathbf{A}}$ is a complex amplitude vector.

a) Specify the complex amplitude vector for waves polarized along the $x$ axis.

b) Specify the complex amplitude vector for waves polarized along the axis angled at 45° between the $x$ and $y$ axes.

In general

$$\tilde{\mathbf{A}} = \tilde{A}_x \hat{\mathbf{x}} + \tilde{A}_y \hat{\mathbf{y}}.$$

where $\tilde{A}_x = A_x e^{i\delta_x}$ and $\tilde{A}_y = A_y e^{i\delta_y}$ with $A_x, A_y$ and $\delta_x, \delta_y$ all real.

c) Suppose that $\tilde{A}_x = Ae^{-i\pi/2}$ and $\tilde{A}_y = A$ with $A > 0$. Determine a real expression for this wave. Sketch the vector describing the wave at $z = 0$ as time passes.

Answer: a) $\tilde{\vec{A}} = Ae^{i\delta_x}\hat{x}$ where $A$ is real

$\Rightarrow \tilde{\vec{f}}(z,t) = Ae^{i\delta_x}e^{i(kz-\omega t)}\hat{x}$

$\Rightarrow \vec{f}(z,t) = A\cos(kz-\omega t+\delta_x)\hat{x}$ oscillates along $\hat{x}$

b) $\tilde{\vec{A}} = A\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})e^{i\delta}$

$\Rightarrow \tilde{\vec{f}}(z,t) = A\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})e^{i(kz-\omega t+\delta)}$

$\Rightarrow \vec{f}(z,t) = A\cos(kz-\omega t+\delta)\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$ oscillates along $\hat{x}+\hat{y}$

c) $\vec{f}(z,t) = \tilde{\vec{A}}e^{i(kz-\omega t)}$

$= [\tilde{A}_x\hat{x} + \tilde{A}_y\hat{y}]e^{i(kz-\omega t)}$

$= (Ae^{-i\pi/2}\hat{x} + A\hat{y})e^{i(kz-\omega t)}$

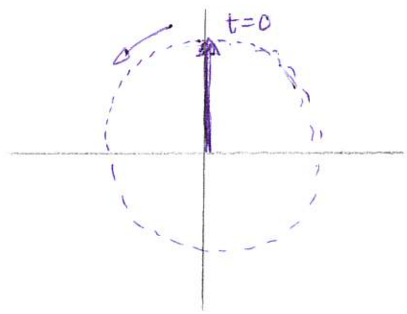
$= A[e^{i(kz-\omega t-\pi/2)}\hat{x} + e^{i(kz-\omega t)}\hat{y}]$

$\Rightarrow \vec{f}(z,t) = A\cos(kz-\omega t-\pi/2)\hat{x} + A\cos(kz-\omega t)\hat{y}$

$$\Rightarrow \qquad \vec{f}(z,t) = A\sin(kz-\omega t)\,\hat{x} + A\cos(kz-\omega t)\,\hat{y}$$

Observe this at $z=0$. Then

$$\vec{f}(0,t) = -A\sin(\omega t)\,\hat{x} + A\cos(\omega t)\,\hat{y}$$

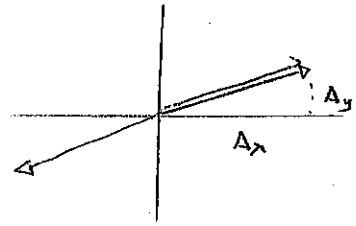



This appears to rotate at a constant rate counter clockwise. The vector traces a circle and is called circularly polarized.

DEMO: PSU-S EM waves circular polarization video

1 Certain broad classes are:

$$\tilde{\vec{A}} = \vec{A} e^{i\delta} = (A_x \hat{x} + A_y \hat{y}) e^{i\delta}$$




and this will oscillate along one line. This is linearly polarized. But if

$$\tilde{\vec{A}} = A_x e^{i\delta_x} \hat{x} + A_y e^{i\delta_y} \hat{y}$$

where $\delta_x \neq \delta_y$ then the vector $\vec{A}$ will trace out an ellipse. This is called ellipitcally polarized..