Thurs: Read 9.1.3,

Fri: HW 5pm

Tues: Class Exam I covers Ch 4, 6, 8.1

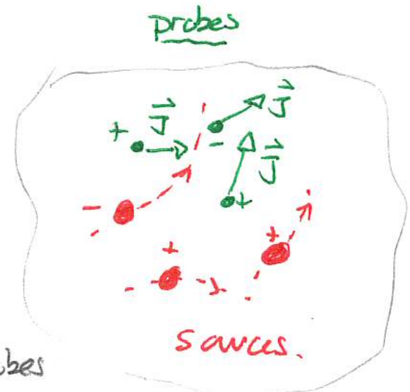
Review: 2016, 2021 class exam 1 all Q

Energy in electromagnetism

Consider a collection of otherwise isolated source charges and currents. We can mentally divide the charges into sources and probes.

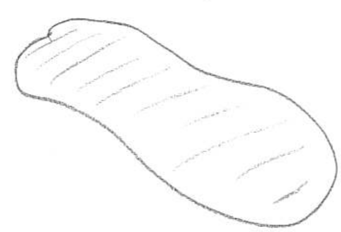
Then the sources:

* produce electric and magnetic fields
* the fields exert forces on the probes
* the fields do work on the probes
* the work changes the kinetic energy of the probes



The division between probes and sources is arbitrary and by considering each charge, at some point, as a probe we can calculate the work done on all the charges.

Then consider a region R. The rate at which work is done on the charges in this region is



$$\frac{dW_{mech}}{dt} = \int_R \vec{E}\cdot\vec{J}\, d\tau'$$

and this gives the rate of change of kinetic energy in the region K_R as

$$\frac{dK_R}{dt} = \int_R \vec{E}\cdot\vec{J}(\vec{r}')\, d\tau'$$

Here $J(\vec{r}')$ is the current density resulting from the charge distribution via $\vec{J}(\vec{r}') = \rho(\vec{r}')\vec{v}(\vec{r}')$ where $\vec{v}(\vec{r}')$ is the velocity of the charge distribution.

Then $\vec{E}(\vec{r}')$ is the electric field in the region, produced by all charges inside and outside the region.

The potential energy stored in the region is

$$U_R = \frac{\epsilon_0}{2}\int_R \vec{E}\cdot\vec{E}\, d\tau' + \frac{1}{2\mu_0}\int_R \vec{B}\cdot\vec{B}\, d\tau'$$

We can then show that:

$$\frac{d}{dt}\left(U_R + K_R\right) = -\oint_S \vec{S}\cdot d\vec{a}$$

where S is the surface of the region and

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

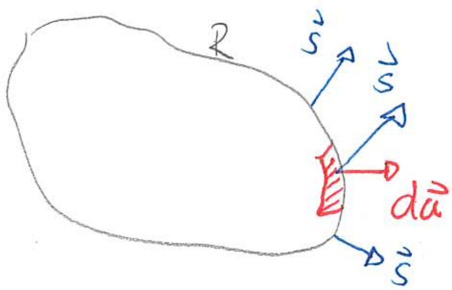
is the Poynting vector.

Poynting's theorem

This is a form of energy conservation that describes:

1) the rate of change of energy $U_R + K_R$ in the region.
2) the rate at which energy leaves the surface with respect to time and area.

Consider a closed surface with an outward area vector $d\vec{a}$. Two possible situations are



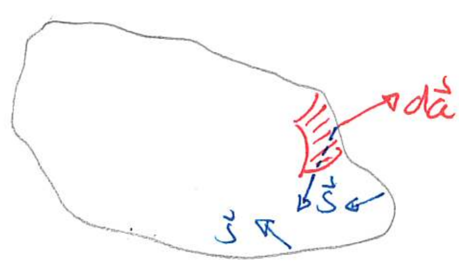
$\vec{S}\cdot d\vec{a} > 0$

gives negative contribution to

$-\oint \vec{S}\cdot d\vec{a}$

$\Rightarrow$ $U_R + K_R$ decreases

$\Rightarrow$ energy leaves region



$\vec{S}\cdot d\vec{a} < 0$

gives positive contribution to

$-\oint \vec{S}\cdot d\vec{a}$

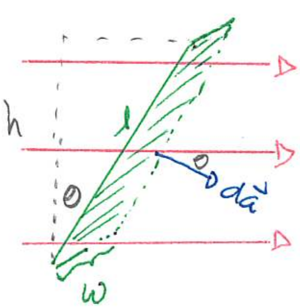
$\Rightarrow$ $U_R + K_R$ increases

$\Rightarrow$ energy enters region

Thus

$\boxed{\text{The Poynting vector } \vec{S} \text{ indicates a direction of energy flow}}$

To make this more precise, consider a region with uniform Poynting vector, and a plane surface at angle θ as illustrated. Then



$$\int \vec{S}\cdot d\vec{a} = S\, da \cos\theta$$
$$= S\, l w \cos\theta$$

But $l \cos\theta = h$ gives

$$\int \vec{S}\cdot d\vec{a} = S \underbrace{hw}_{\text{area of perpendicular surface}}$$

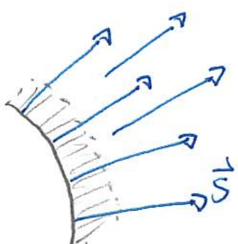
This is the same as the integral over a surface perpendicular to $\vec{S}$

Thus we see that

$$\underbrace{S A}_{\substack{\text{area of surface}\\ \text{perpendicular to } \vec{S}}} = \text{rate at which energy flows}$$

Thus

$$\boxed{\vec{S} \text{ quantifies the rate at which energy would flow, per unit area, through a surface perpendicular to } \vec{S}.}$$



$\hookrightarrow$ perpendicular surface, area A

$\Rightarrow$ energy per second through surface is

SA.

The units of Poynting vector are obtained by

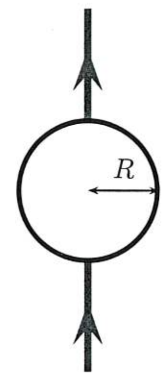
$$\left.\begin{aligned} E &\sim N/C \\ B &\sim \frac{N}{A\cdot m} \\ \mu_0 &= N/A^2 \end{aligned}\right\} \Rightarrow S \sim \frac{A^{\cancel{2}}}{\cancel{N}} \cdot \frac{\cancel{N}}{C} \cdot \frac{N}{\cancel{A}\cdot m} = \frac{\cancel{C}\,N}{s\,\cancel{C}\cdot m} = \frac{Nm}{m^2 s}$$

$$= J/m^2 s.$$

These are units of energy per second per area.

1 Poynting vector and energy flow

A small spherical shell with radius R carries charge $q > 0$ which is uniformly distributed. Two wires carry constant currents in the indicated directions. The wires are very long compared to the sphere radius.



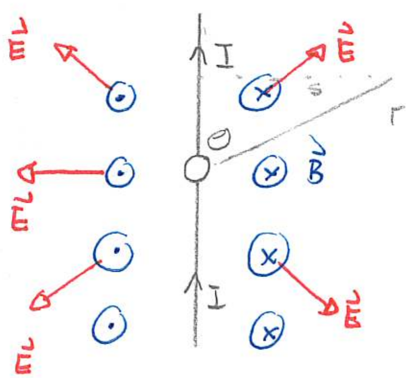
a) Sketch the electric and magnetic fields produced by charge and current distributions.

b) Determine the Poynting vector associated with these fields.

c) In which general sense does energy flow in this situation. Does this match what would be expected in terms of any mechanical work done?

d) Determine the rate at which energy flows through the entire horizontal plane through the middle of the sphere.

Answer: a)



b) By Gauss' Law

$$\vec{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{r} & r > R \\ 0 & r < R \end{cases}$$

By Ampere's Law

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

where s is the radial distance from the wire. Then $\vec{S} = 0$ inside

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\frac{\mu_0 I}{2\pi s}\underbrace{\hat{r}\times\hat{\phi}}_{-\hat{\theta}} = -\frac{qI}{8\pi^2\epsilon_0 r^2 s}\hat{\theta}$$

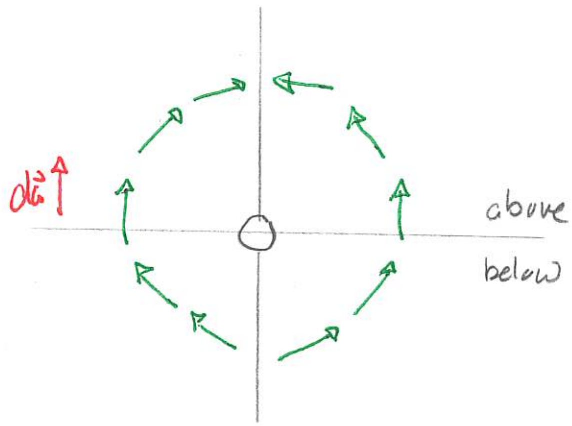
But

$$s = r\cos\theta$$

gives:

$$\vec{S} = \begin{cases} 0 & \text{inside} \\ -\dfrac{qI}{8\pi^2\epsilon_0 r^3 \sin\theta}\,\hat{\theta} & \text{outside.} \end{cases}$$

c)



It tends to flow from below to above. In the region above the wire

$$\vec{J}\cdot\vec{E} > 0$$

and below

$$\vec{J}\cdot\vec{E} < 0$$

So kinetic energy is moving from below to above.

d) Integrate over the surface

$$\left.\begin{array}{l} R \le r' \le \infty \\ 0 \le \phi' \le 2\pi \\ \theta' = \pi/2 \end{array}\right\} \Rightarrow d\vec{a} = -r' \overset{\pi/2}{\sin\theta}\, dr' d\phi' \hat{\theta}' = -r'\, dr' d\phi' \hat{\theta}'$$

$$\Rightarrow \vec{S}\cdot d\vec{a} = \frac{qI}{8\pi^2\epsilon_0 r'^3 \sin\pi/2}\, r' dr' d\phi' = \frac{qI}{8\pi^2\epsilon_0 r'^2}\, dr' d\phi'$$

Then

$$\oint \vec{S}\cdot d\vec{a} = \frac{qI}{8\pi^2\epsilon_0} \underbrace{\int_R^\infty \frac{1}{r'^2}\, dr'}_{1/R} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} = \frac{qI}{4\pi\epsilon_0 R}$$

Differential form of Poynting's theorem

Poyntings theorem is :

$$\frac{d}{dt}\left\{\frac{\epsilon_0}{2}\int_R \vec{E}\cdot\vec{E}\,d\tau' + \frac{1}{2\mu_0}\int \vec{B}\cdot\vec{B}\,d\tau' + \int \vec{E}\cdot\vec{J}\,d\tau'\right\} = -\oint \vec{S}\cdot d\vec{a}$$

$$= -\int_R \vec{\nabla}\cdot\vec{S}\,d\tau'$$

Thus

$$\int_R \left[\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}\right) + \vec{E}\cdot\vec{J}\right] d\tau' = -\int_R \vec{\nabla}\cdot\vec{S}\,d\tau'$$

This is true for any region. Thus:

$$\frac{\partial u}{\partial t} + \vec{E}\cdot\vec{J} = -\vec{\nabla}\cdot\vec{S}$$

where

$$u = \frac{\epsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}$$

is the energy density.

This is a differential version of the Poynting theorem.

Electromagnetic Waves

We will see that, in a vacuum region, Maxwell's equations can be manipulated to provide wave equations. These will have:

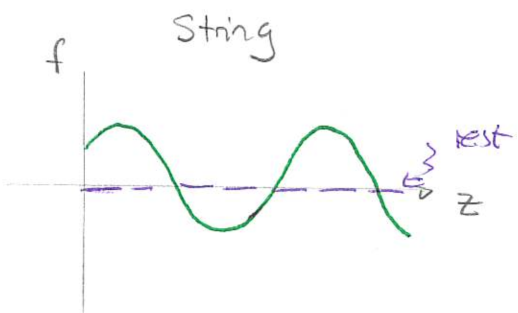
1) the form of the classical wave equation
2) a vector nature.

Wave equation in one dimension

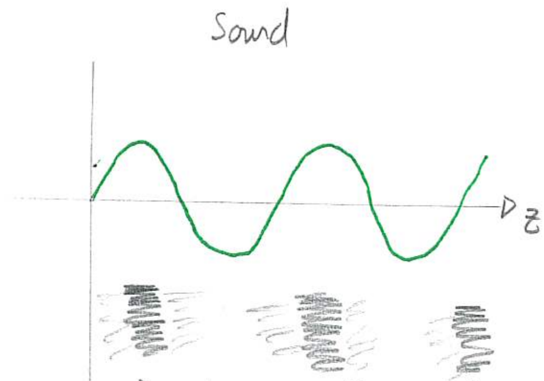
We review the basic aspects of classical waves by considering a generic wave along one spatial dimension. Mathematically this will be a function of

1) one spatial variable z
2) time t

The generic function will be describe as $f(z,t)$. Snapshots of examples are:



$f \equiv$ vertical displacement from rest



$f =$ density difference from normal

Such wave equations appear when applying basic physics such as classical mechanics or thermodynamics.

The equation that emerges has the form:

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

where v is a constant, called the wave speed.

A solution to this equation is a function $f(z,t)$ that satisfies the equation at all times and locations. The theory of differential equations provides:

> The general solution to
>
> $$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$
>
> is
>
> $$f(z,t) = g(z-vt) + h(z+vt)$$
>
> where $g(u)$ and $h(u)$ are any suitably differentiable functions of a single variable.

Proof: Consider $f(z,t) = g(z-vt)$. Then with $u = z-vt$

$$\frac{\partial f}{\partial z} = \left.\frac{dg}{du}\right|_{u=z-vt}\frac{\partial u}{\partial z} = \left.\frac{dg}{du}\right|_{z-vt}$$

$$\frac{\partial^2 f}{\partial z^2} = \left.\frac{d^2 g}{du^2}\right|_{u=z-vt}\frac{\partial u}{\partial z} = \left.\frac{d^2 g}{du^2}\right|_{u=z-vt}$$

$$\frac{\partial f}{\partial t} = \left.\frac{dg}{du}\right|_{u=z-vt}\frac{\partial u}{\partial t} = -v\left.\frac{dg}{du}\right|_{u=z-vt}$$

$$\frac{\partial^2 f}{\partial t^2} = -v\left.\frac{d^2 g}{du^2}\right|_{u=z-vt}\frac{\partial u}{\partial t} = v^2\left.\frac{d^2 g}{du^2}\right|_{u=z-vt} = v^2\frac{\partial^2 f}{\partial z^2}$$

A similar derivation applies to $f(z,t) = h(z+vt)$

2 One dimensional wave equation solutions

The one dimensional wave equation is

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$

Assume that $v > 0$.

a) Show that

$$f(z,t) = A\sin\left(kz - \omega t\right)$$

where k, ω and A are constants, is a solution provided that k, ω and v satisfy a particular relationship.

b) By considering the location of the maximum of

$$f(z,t) = Be^{-(z-vt)^2/a^2}$$

describe the direction of propagation and the speed with which the disturbance travels.

c) By considering the location of the maximum of

$$f(z,t) = Be^{-(z+vt)^2/a^2}$$

describe the direction of propagation and the speed with which the disturbance travels.

Answer: a)

$$\frac{\partial f}{\partial z} = kA\cos(kz-\omega t) \qquad \frac{\partial f}{\partial t} = -\omega A\cos(kz-\omega t)$$

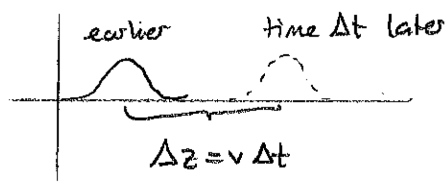
$$\frac{\partial^2 f}{\partial z^2} = -k^2 A\sin(kz-\omega t) \qquad \frac{\partial^2 f}{\partial t^2} = -\omega A(+\omega)\sin(kz-\omega t) = -\omega^2 A\sin(kz-\omega t).$$

Thus $\displaystyle\frac{\partial^2 f}{\partial z^2} = \frac{k^2}{\omega^2}\frac{\partial^2 f}{\partial t^2}$

Thus $\displaystyle\frac{1}{v^2} = \frac{k^2}{\omega^2} \Rightarrow \boxed{v = \omega/k.}$

b) Max occurs when $(z-vt)^2/a^2 = 0 \Rightarrow z = vt$

travels with speed v to right



c) Max occurs when $(z+vt)^2/a^2 = 0 \Rightarrow z = -vt$

travels with speed v to left.