Fri: HW 5pm

Tues: HW

Read: 8.1, 9.1

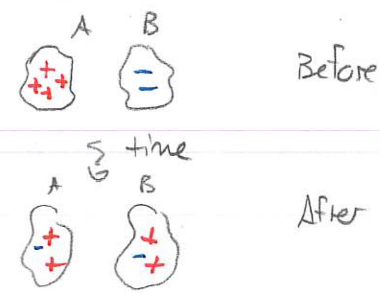
Conservation laws

In physics, conservation laws describe how a quantity, aggregated for an entire system, remains constant as the system evolves. There will be three conservation laws in electromagnetism, dealing with:

1) conservation of charge
2) conservation of energy
3) conservation of momentum. $\Rightarrow$ from Maxwell's equations

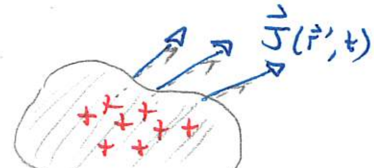
Charge conservation

Charge conservation describes how the total charge in a system remains constant even though the charges on the constituents vary.



In the course of such evolution the rearranging charges will produces currents and the charge in a region must be related to the current entering or leaving. We convert these ideas into statements involving charge and current density. Let

$$\rho(\vec{r}\,',t)$$



represent the charge density in all space. If this changes then it means that some current must flow. Consider a region R enclosed by a surface S. Let

$Q(t)$ = total charge in region at time t.

$I(t)$ = current leaving region at time t

Then an assumption is that

$$I = -\frac{dQ}{dt}$$

But

$$I = \oint_S \vec{J}\cdot d\vec{a}$$

$$Q = \int_R \rho(\vec{r}',t)\,d\tau'$$

gives:

$$\oint_S \vec{J}\cdot d\vec{a} = -\int_R \frac{\partial \rho(\vec{r}',t)}{\partial t}\,d\tau'$$

$$\Rightarrow \quad \int_R \vec{\nabla}\cdot\vec{J}\,d\tau' = -\int_R \frac{\partial \rho(\vec{r}',t)}{\partial t}\,d\tau'$$

This can only be true for all regions R. Thus

Charge is conserved and current results from charge flow	$\Longleftrightarrow$	$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$

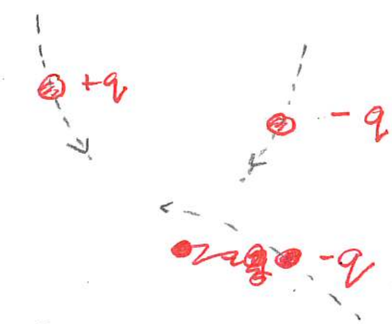
This is the continuity equation and it constrains $\vec{J}$ and ρ in relation to each other. However, it does not completely determine one from the other. For example given $\vec{J}$, this will only allow one to determine ρ up to an overall time dependent piece $\rho_0(\vec{r}')$.

Energy in electromagnetism

We have encountered energy in electrostatics and magnetostatics.

In electrostatics

Assemble charges, eventually at rest, from infinitely far apart.



This requires external forces.

The work done by external forces satisfies

$$W_{ext} = -W_{elec}$$

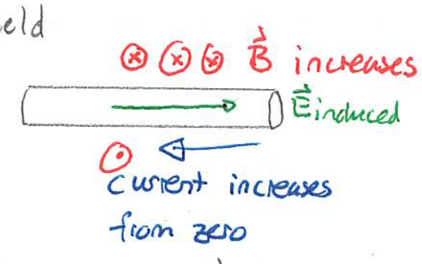
where W_{elec} is the work done by electrostatic forces

Calculating W_{elec} gives W_{ext} which we describe as the energy stored in the electric field

$$\underset{\overset{\|}{W_{ext}}}{U_{elec}} = \frac{\epsilon_0}{2}\int_{\text{all space}} \vec{E}\cdot\vec{E}\, d\tau$$

In magnetostatics

Assembling a current results in an opposing induced electric field



The increasing $\vec{B}$ field produces an induced electric field that opposes the current. This requires external work.

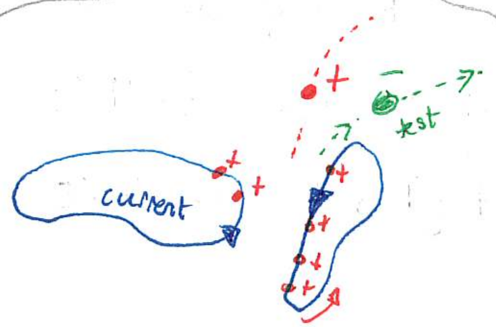
Calculating the work done by the induced field gives the energy stored in the magnetic field

$$U_{mag} = \frac{1}{2\mu_0}\int_{\text{all space}} \vec{B}\cdot\vec{B}\, d\tau$$

This suggests that the energy required to assemble a charge and current distribution, or the energy stored in the electric and magnetic fields is

$$\boxed{U = \frac{\epsilon_0}{2}\int_{\text{all space}} \vec{E}\cdot\vec{E}\, d\tau + \frac{1}{2\mu_0}\int_{\text{all space}} \vec{B}\cdot\vec{B}\, d\tau}$$

We now aim to put this on a firm footing using:



The fields produced by the source currents and charges do mechanical work on the test charge.

W_{mech}.

Relate the rate of change of mechanical work to the rate of change of kinetic energy of the test

$$\frac{dW_{mech}}{dt} \text{ and } \frac{dK}{dt}$$

Consider the energy stored

$$U = \frac{\epsilon_0}{2}\int \vec{E}\cdot\vec{E}\, d\tau' + \frac{1}{2\mu_0}\int \vec{B}\cdot\vec{B}\, d\tau'$$

Use Maxwell's eqns to relate

$$\frac{dU}{dt} \text{ to } \frac{dW_{mech}}{dt}$$

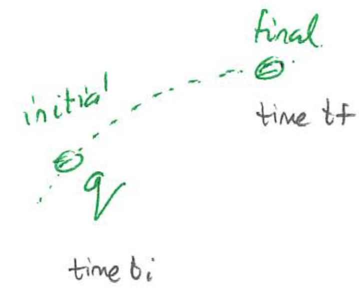
Show

$$\frac{dU}{dt} + \frac{dK}{dt} = 0 \Rightarrow U + K = \text{constant}$$

This is energy conservation.

Mechanical work done by fields on charges

Consider a single test particle moving in the presence of electric and magnetic fields (produced by other sources). Then the mechanical work done on this is



$$W_{mech} = \int_{initial}^{final} \vec{F}\cdot d\vec{l}$$

But

$$\vec{F} = q\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

gives

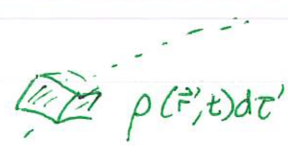
$$W_{mech} = q\left\{\int \vec{E}\cdot d\vec{l} + \int(\vec{v}\times\vec{B})\cdot d\vec{l}\right\}$$

Now $\vec{v}$ is parallel to $d\vec{l}$. Thus $\vec{v}\times\vec{B}\cdot d\vec{l} = 0$. So

$$W_{mech} = q\int \vec{E}\cdot d\vec{l} = q\int_{t_i}^{t} \vec{E}\cdot\vec{v}\,dt' \qquad \text{since } d\vec{l} = \vec{v}dt'.$$

Now consider a more general charge distribution, that is not just a single test charge. Then we replace

$$q \to \rho(\vec{r}',t)d\tau'$$



and $\vec{E}$ is the field produced by all charges and currents. Then

$$W_{mech} = \int d\tau'\int \rho(\vec{r}',t)\vec{E}\cdot\vec{v}\,dt'$$

$$= \int d\tau'\int dt\,\vec{E}\cdot\vec{v}\rho(\vec{r}',t)$$

But the current density is $\vec{J} = \rho(\vec{r}',t)\vec{v}$

Thus

$$W_{mech} = \int_{t_i}^{t} dt \int d\tau' \; \vec{J}(\vec{r}\,',t') \cdot \vec{E}(\vec{r}\,',t')$$

Then

$$\boxed{\frac{dW_{mech}}{dt} = \int_{\text{all space}} d\tau' \; \vec{J}(\vec{r}\,',t) \cdot \vec{E}(\vec{r}\,',t)}$$

Now let the kinetic energy be $K(t)$. We have

$$W_{mech}(t) = K(t) - K(t_i)$$

So

$$\frac{dW_{mech}}{dt} = \frac{dK}{dt}$$

It follows that

> The rate at which the fields do mechanical work on the charge distribution is
>
> $$\frac{dW_{mech}}{dt} = \int_{\text{all space}} \vec{J} \cdot \vec{E} \, d\tau'$$
>
> and the rate of change of kinetic energy of the charges in the distribution is
>
> $$\frac{dK}{dt} = \int_{\text{all space}} \vec{J} \cdot \vec{E} \, d\tau'$$

Note that the rate at which the fields do work is the <u>power delivered</u>

$$P = \frac{dW_{mech}}{dt} \Rightarrow P = \int_{\text{all space}} \vec{J} \cdot \vec{E} \, d\tau$$

1 Power to sustain a uniform current in a cylindrical wire

A current with uniform density flows in the axial direction along a cylindrical wire with radius R. The current is sustained by an electric field which points in the direction of the current and is uniform.

a) Determine an expression for the potential difference between the ends of the wire in terms of the electric field and the length of the wire.

b) Using

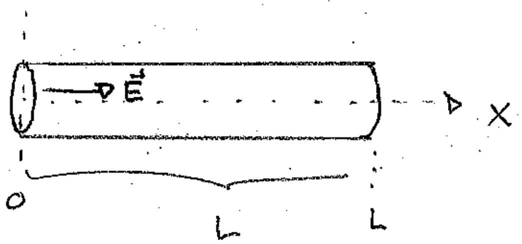
$$\frac{dW_{\text{mech}}}{dt} = \int \mathbf{E} \cdot \mathbf{J}\, d\tau'$$

for this situation, show that

$$P = I\Delta V$$

where I is the current that flows along the wire.

Answer a)



$$\Delta V = -\int \vec{E}\cdot d\vec{l} \implies V(L) - V(0) = -\int \vec{E}\cdot d\vec{l}$$

Then $d\vec{l} = dx\,\hat{x}$ $\quad \vec{E} = E\hat{x} \implies \vec{E}\cdot d\vec{l} = E\,dx$

$$\implies V(L) - V(0) = -EL$$

Let $\Delta V = V(0) - V(L) \implies \Delta V = EL$

b) $$P = \frac{dW_{mech}}{dt} = \int \vec{E}\cdot\vec{J}\,d\tau'$$

Then $\vec{J} = J\hat{x}$ and $J = I/\text{cross sectional area} = \dfrac{I}{\pi R^2}$

Thus

$$P = \int E\hat{x}\cdot J\hat{x}\,d\tau' = \frac{EI}{\pi R^2}\int d\tau' = \frac{EI}{\pi R^2}\pi R^2 L = ELI$$
$$= \Delta V I$$

$$\implies P = I\Delta V$$

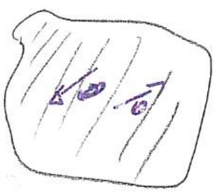
Energy Conservation

Generally energy conservation can take the forms:

1) all space - consider the mechanical work done on all charges
   - consider the energy stored in the fields in all space
   - energy is conserved if $\frac{dU}{dt} + \frac{dW_{mech}}{dt} = 0$

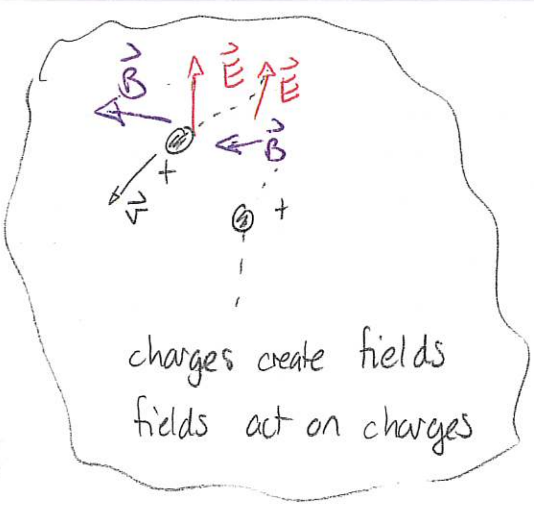
$$\Rightarrow \frac{dU}{dt} + \frac{dK}{dt} = 0$$

2) restricted region - consider mechanical work done on all charges in some region, W_{mech}
   - consider energy stored in some region U
   - then



$$\frac{dU}{dt} + \frac{dW_{mech}}{dt} = \text{rate of energy flow out of region}$$

Poynting's theorem will deal with both cases, and the second is more general. As a limited case we consider the first. In this situation



Calculate total energy and rate of change of total energy in fields

→ Compare to work done

2 Poynting's theorem: all space

The energy stored in the fields is

$$U = \frac{\epsilon_0}{2} \int \mathbf{E} \cdot \mathbf{E} \, \mathrm{d}\tau + \frac{1}{\mu_0 2} \int \mathbf{B} \cdot \mathbf{B} \, \mathrm{d}\tau.$$

Assume that the charges and currents that produce the fields are localized. Differentiate the energy with respect to time, substitute from Maxwell's equations and use vector calculus identities to show that

$$\frac{\partial U}{\partial t} = -\int \mathbf{E} \cdot \mathbf{J} \, \mathrm{d}\tau$$

provided that integration is done over all space.

Answer:

$$\frac{dU}{dt} = \frac{\epsilon_0}{2} \int \frac{\partial \vec{E}}{\partial t} \cdot \vec{E} \, d\tau + \frac{\epsilon_0}{2} \int \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \, d\tau$$

$$+ \frac{1}{2\mu_0} \int \frac{\partial \vec{B}}{\partial t} \cdot \vec{B} \, d\tau + \frac{1}{2\mu_0} \int \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} \, d\tau.$$

$$= \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot \vec{E} \, d\tau + \frac{1}{\mu_0} \int \frac{\partial \vec{B}}{\partial t} \cdot \vec{B} \, dt.$$

integrals over all space

Maxwell's equations give:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Rightarrow \quad \frac{\partial \vec{E}}{\partial t} = \frac{1}{\mu_0 \epsilon_0} \vec{\nabla} \times \vec{B} - \frac{1}{\epsilon_0} \vec{J}$$

Thus

$$\frac{dU}{dt} = \frac{\epsilon_0}{\mu_0 \epsilon_0} \int_{\text{all space}} (\vec{\nabla} \times \vec{B}) \cdot \vec{E} \, d\tau - \int_{\text{all space}} \vec{J} \cdot \vec{E} \, d\tau - \frac{1}{\mu_0} \int_{\text{all space}} (\vec{\nabla} \times \vec{E}) \cdot \vec{B} \, d\tau$$

$$= \frac{1}{\mu_0} \int_{\text{all space}} \left[(\vec{\nabla} \times \vec{B}) \cdot \vec{E} - (\vec{\nabla} \times \vec{E}) \cdot \vec{B} \right] d\tau - \int_{\text{all space}} \vec{J} \cdot \vec{E} \, d\tau$$

Now an identity is

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$$

Thus

$$\frac{dU}{dt} = -\frac{1}{\mu_0} \int_{\text{all space}} \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \, d\tau - \int_{\text{all space}} \vec{J} \cdot \vec{E} \, d\tau$$

The divergence theorem gives:

$$\frac{dU}{dt} = -\frac{1}{\mu_0} \oint_{\text{surface}} (\vec{E} \times \vec{B}) \cdot d\vec{a} - \int_{\text{all space}} \vec{J} \cdot \vec{E} \, d\tau$$

Now if $\vec{E} \times \vec{B} \to \frac{1}{r^3}$ or faster as $r \to \infty$ the surface integral goes to zero. Thus

$$\frac{dU}{dt} = -\int \vec{E} \cdot \vec{J} \, d\tau$$

$$= -\frac{dW_{mech}}{dt}$$

To summarize:

Given that $\vec{E}, \vec{B}$ are produced by localized charge and current distributions then the total energy

$$U = \frac{\epsilon_0}{2}\int_{\text{all space}} \vec{E}\cdot\vec{E}\, d\tau + \frac{1}{2\mu_0}\int_{\text{all space}} \vec{B}\cdot\vec{B}\, d\tau$$

satisfies:

$$\frac{dU}{dt} + \frac{d W_{mech}}{dt} = 0$$

where the mechanical work done by the fields on the charge distribution satisfies

$$\frac{dW_{mech}}{dt} = \int \vec{J}\cdot\vec{E}\, d\tau$$

Then

$$\frac{dW_{mech}}{dt} = \frac{dK}{dt}$$

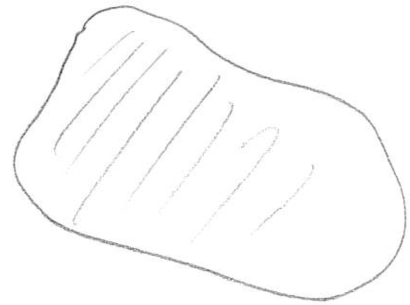
where K is the total kinetic energy of the matter implies:

$$U + K = \text{constant}$$

This is the conservation of energy.

Energy in a region of space

We can relax the requirement that the region of integration be all space. We then have for region R



region R

Definition (energy stored in region)

$$U = \frac{\epsilon_0}{2}\int_R \vec{E}\cdot\vec{E}\,d\tau + \frac{1}{2\mu_0}\int_R \vec{B}\cdot\vec{B}\,d\tau$$

Maxwell's equations

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\mu_0\epsilon_0}\vec{\nabla}\times\vec{B} - \frac{1}{\epsilon_0}\vec{J}$$

Vector calculus

$$\frac{dU_{region}}{dt} = -\frac{1}{\mu_0}\oint_S (\vec{E}\times\vec{B})\cdot d\vec{a} - \int_R \vec{J}\cdot\vec{E}\,d\tau$$

Then with the usual interpretation of the last integral we get

$$\frac{dU_{region}}{dt} + \frac{dK_{region}}{dt} = -\frac{1}{\mu_0}\int(\vec{E}\times\vec{B})\cdot d\vec{a}$$

$$\Rightarrow \boxed{\underbrace{\frac{d}{dt}\left(U_{region} + K_{region}\right)}_{\text{rate at which energy in region changes}} = -\frac{1}{\mu_0}\int(\vec{E}\times\vec{B})\cdot d\vec{a}}$$

Define the Poynting vector

$$\boxed{\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})} \Rightarrow \boxed{\frac{d}{dt}\left(U_{region} + K_{region}\right) = -\oint_{\text{surface of region}} \vec{S}\cdot d\vec{a}}$$