Tues: HW by 5pm
Thurs Read 8.1
Fri: HW by 5pm

DEMO: Levitating frog - Diamagnetic water.

Maxwell's equations in matter

We have shown that in electrostatics, the free charges (partly) determine the electric displacement via $\vec{\nabla}\cdot\vec{D} = \rho_f$. The free currents partly determine the auxiliary magnetic field via $\vec{\nabla}\times\vec{H} = \vec{J}_f$. We want to determine how these unfold in time-varying situations. The basis will be Maxwells equations

> Given total charge density $\rho(\vec{r}')$ (includes free and bound charge) and total current density $\vec{J}(\vec{r}')$ the fields produced by these sources satisfy
>
> $$\vec{\nabla}\cdot\vec{E} = \rho(\vec{r}')/\epsilon_0$$
> $$\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$$
> $$\vec{\nabla}\cdot\vec{B} = 0$$
> $$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Solving the resulting differential equations gives $\vec{E}$ and $\vec{B}$.
Now consider the situation where matter is present. First the bound charge contributions are described via

Polarization $\vec{P}$

Electric displacement
$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$

Then electric displacement is determined by
$$\vec{\nabla}\cdot\vec{D} = \rho_f$$
$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$$
where ρ_f is the free charge only

The magnetic contributions provided by the material are described by:

Magnetization
$\vec{M}$

Auxilary fied
$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

The auxiliary field satisfies:

$$\vec{\nabla} \times \vec{H} = \vec{J}_f$$

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$$

where $\vec{J}_f$ is the free current density

Note that we obtain the free charge density via:

Bound charge

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

Free charge

$$\rho_f = \rho - \rho_b = \rho + \vec{\nabla} \cdot \vec{P}$$

Band current

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

Free current

$$\vec{J}_f = \vec{J} - \vec{J}_b = \vec{J} - \vec{\nabla} \times \vec{M}$$

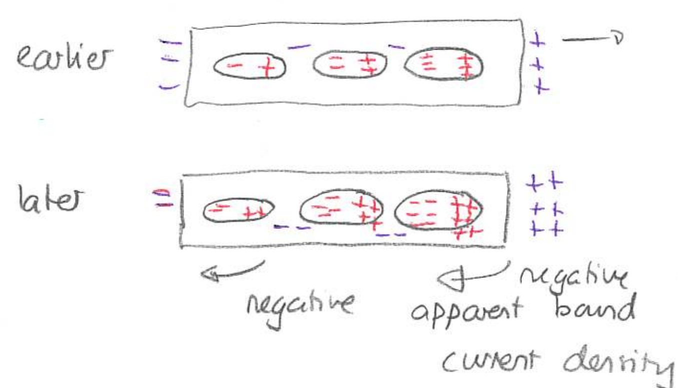
These refered to time-independent situations Consider a situation where the polarization is time-dependent:

$$\vec{P} = xt\,\hat{x}$$

Then

$$\rho_b = -xt$$

and there will be an apparent flow of band charge as time passes.



We focus on modifying

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

First $\vec{B} = \mu_0 \vec{H} + \vec{M}\mu_0 \Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \times \vec{H} + \vec{\nabla} \times \vec{M}\mu_0$

Second $\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P}) \Rightarrow \frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon_0}\left[\frac{\partial \vec{D}}{\partial t} - \frac{\partial \vec{P}}{\partial t}\right]$

Substitution gives:

$$\mu_0 \vec{\nabla} \times \vec{H} + \vec{\nabla} \times \vec{M}\mu_0 = \mu_0 \vec{J} + \cancel{\epsilon_0}\mu_0 \frac{1}{\cancel{\epsilon_0}}\left[\frac{\partial \vec{D}}{\partial t} - \frac{\partial \vec{P}}{\partial t}\right]$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{H} = \vec{J} - \vec{\nabla} \times \vec{M} + \frac{\partial \vec{D}}{\partial t} - \frac{\partial \vec{P}}{\partial t}$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{H} = \vec{J} - \vec{J}_b - \frac{\partial \vec{P}}{\partial t} + \frac{\partial \vec{D}}{\partial t}$$

We define the <u>polarization current</u> as

$$\boxed{\vec{J}_p = \frac{\partial \vec{P}}{\partial t}}$$

Note that this is associated with the change in time over bound charges and can be consider as a contribution to the bound current that does not arise from the magnetization. Then if we redefine the free current density as

$$\vec{J}_f = \vec{J} - \vec{J}_b - \vec{J}_p$$

we get:

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

This leaves <u>Maxwell's equations in matter</u>:

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho_f \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{H} &= \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \end{aligned}}$$

The scheme now is

Given free charge density ρf and free current density $\vec{J}$

Given relationships between
* $\vec{D}$ and $\vec{E}$
* $\vec{H}$ and $\vec{B}$

Solve

$$\vec{\nabla}\cdot\vec{D} = \rho f \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

for $\vec{D}$ and $\vec{H}$

Combine solutions and relationships to get $\vec{E}$ and $\vec{B}$

For <u>linear homogeneous media</u> the relationships between $\vec{D}$ and $\vec{E}$ and $\vec{H}$ and $\vec{B}$ are:

$$\vec{E} = \vec{D}/\epsilon$$

$$\vec{B} = \mu\vec{H}$$

where ϵ, μ are independent of position. Then these give

$$\boxed{\begin{aligned}
\vec{\nabla}\cdot\vec{D} &= \rho f \\
\vec{\nabla}\times\vec{D} &= -\epsilon\mu\frac{\partial\vec{H}}{\partial t} \\
\vec{\nabla}\cdot\vec{H} &= 0 \\
\vec{\nabla}\times\vec{H} &= \vec{J}_f + \frac{\partial\vec{D}}{\partial t}
\end{aligned}}$$

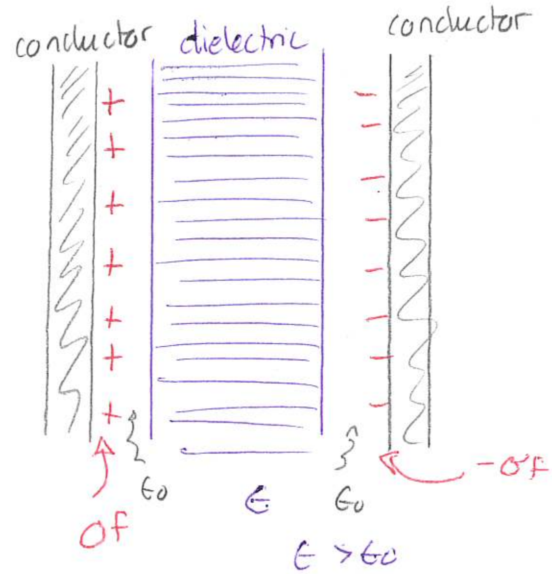
We can solve these for $\vec{D}, \vec{H}$.

Boundary conditions.

There are situations with discontinuous jumps in material properties. These will

* produce jumps in fields
* result in surface currents and charges.

We aim to relate these. As an example consider an electrostatic situation involving a parallel plate capacitor with a dielectric slab between and parallel to the plates.



Assume the indicated uniform free charge densities. Then

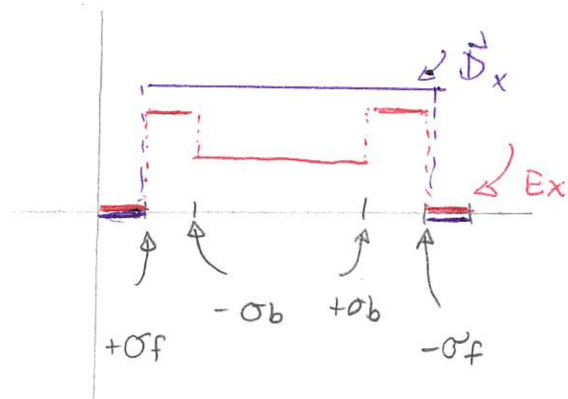
1) inside the conductor plates

$$\vec{D} = \vec{E} = 0$$

2) the free charge resides on the inner surfaces of the plates

3) between the plates

$$\vec{D} = \sigma_f \hat{x}$$



4) between the plates

$$\vec{E} = \begin{cases} \sigma_f/\epsilon_0 & \text{outside dielectric} \\ \sigma_f/\epsilon & \text{inside dielectric} \end{cases}$$

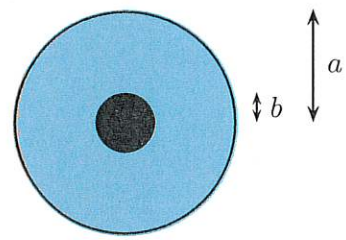
Note that the field components display discontinuities:

a) jumps in electric displacement $\vec{D}$ are associated with sheets of free charge

b) " " electric field $\vec{E}$ " " " sheets of bound or free charge.

1 Boundary conditions for electric displacement and fields

An infinite conducting cylindrical shell of radius a is concentric with an infinite conducting rod of radius $b < a$. The region between the conductors is filled with a linear dielectric with constant ϵ. Suppose that the surface charge density on the inner conductor is uniform and $\sigma_f > 0$.



a) Determine the electric displacement and electric field, in terms of σ_f, for all ~~$S < a$~~. $s < a$

b) Determine an expression for the change in the perpendicular component of the electric displacement across the inner boundary ($s = b$).

c) Determine an expression for the change in the tangential component of the electric displacement across the inner boundary.

d) Determine an expression for the change in the perpendicular component of the electric field across the inner boundary in terms of the free charge density on that surface.

e) Determine an expression for the change in the tangential component of the electric field across the inner boundary.

f) Determine an expression for the bound surface charge on the inner surface of the dielectric in terms of the free charge density on the neighboring conductor. Determine the total charge density at the inner interface.

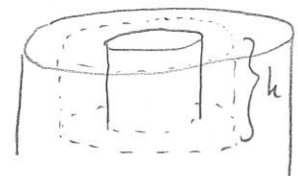
g) Determine an expression for the change in the perpendicular component of the electric field across the inner boundary in terms of the total charge density at the inner interface.

Answer: a) By symmetry $\vec{D}(\vec{r}) = D_s(s)\,\hat{s}$. We use Gauss' Law

$$\oint \vec{D}\cdot d\vec{a} = q_{\text{free enc}}$$

The surface is a cylinder of radius s and height h. Then, for

$$\oint \vec{D}\cdot d\vec{a}, \quad \left.\begin{array}{c} s' = s \\ 0 \leq \phi' \leq 2\pi \\ 0 \leq z' \leq h \end{array}\right\} \quad d\vec{a} = s\,d\phi'\,dz'\,\hat{s}$$



$$\Rightarrow \oint \vec{D}\cdot d\vec{a} = \int_0^h dz' \int_0^{2\pi} d\phi'\; s D_s(s) = 2\pi h s\, D_s(s)$$

Then

$$q_{\text{free enc}} = \begin{cases} 2\pi b h \sigma_F & s > b \\ 0 & s < b \end{cases}$$

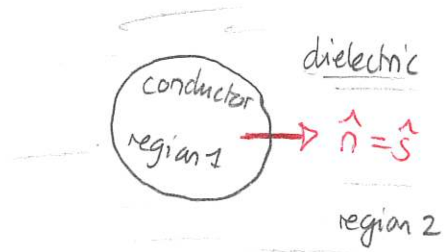
$$\oint \vec{D}\cdot d\vec{a} \Rightarrow \cancel{2\pi}\cancel{h}\, s\, D_s(s) = \begin{cases} \cancel{2\pi}\, b \cancel{h}\, \sigma_F & s > b \\ 0 & s < b \end{cases}$$

$$\Rightarrow \quad \boxed{\vec{D} = \begin{cases} \dfrac{b\,\sigma_f}{s}\,\hat{s} & b < s < a \\ 0 & s < b \end{cases}}$$

Then the electric field is

$$\boxed{\vec{E} = \begin{cases} 0 & s < b \\ \dfrac{b\,\sigma_f}{\epsilon s}\,\hat{s} & b < s < a \end{cases}}$$

b) The perpendicular component of $\vec{D}$ is decided by the normal from the interior conductor to the dielectric. Then



inside the conductor the perpendicular component is

$$D_{1\perp} = \vec{D}\cdot\hat{s} = 0$$

inside the dielectric the perpendicular component is

$$D_{2\perp} = \vec{D}\cdot\hat{s} = \frac{b\,\sigma_f}{s}\Big|_{\text{evaluated as } s=b}$$

$$\Rightarrow \quad \begin{aligned} D_{1\perp} &= 0 \\ D_{2\perp} &= \sigma_f \end{aligned} \qquad \Rightarrow \qquad \boxed{D_{2\perp} - D_{1\perp} = \sigma_f}$$

c) The tangential components $(\vec{D} - (\vec{D}\cdot\hat{s})\hat{s})$ are both zero:

$$\vec{D}_{2\parallel} = \vec{D}_{1\parallel} = 0 \qquad \Rightarrow \qquad \vec{D}_{2\parallel} = \vec{D}_{1\parallel}$$

d) $E_\perp = \vec{E}\cdot\hat{s}$ and

Inside the conductor $\quad E_{1\perp} = 0$

outside the conductor (at $b=s$) $\quad E_{2\perp} = D_{2\perp}/\epsilon$

$$\Rightarrow \quad \boxed{\epsilon E_{2\perp} - E_{1\perp} = \sigma_f}$$

e) Since $\vec{E}_{1\parallel} = \vec{E}_{2\parallel} = 0$ we get $\boxed{\vec{E}_{2\parallel} = \vec{E}_{1\parallel}}$

f) The bound surface charge density is

$$\sigma_b = \vec{P} \cdot \hat{n}$$

where $\hat{n}$ is the outward normal from the inner surface. Thus $\hat{n} = -\hat{s}$ and

$$\sigma_b = -\vec{P} \cdot \hat{s}$$

But $\quad \vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad \Rightarrow \quad \vec{P} = \vec{D} - \epsilon_0 \vec{E}$

$$= \epsilon \vec{E} - \epsilon_0 \vec{E}$$

$$= (\epsilon - \epsilon_0) \vec{E}$$

So $\quad \sigma_b = -(\epsilon - \epsilon_0) \vec{E} \cdot \hat{s}$

$$= -(\epsilon - \epsilon_0) \frac{b\sigma_f}{\epsilon s}, \quad s = b$$

$$= \left[\frac{\epsilon_0}{\epsilon} - 1\right] \sigma_f$$

At this interface the total charge density is

$$\sigma = \sigma_f + \sigma_b = \frac{\epsilon_0}{\epsilon} \sigma_f \quad \Rightarrow \quad \sigma = \sigma_f (\epsilon_0/\epsilon).$$

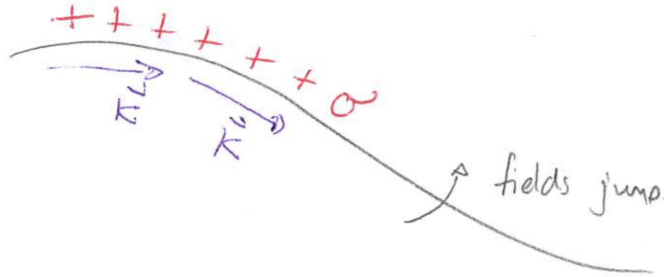
g) we have

$$\epsilon E_{2\perp} - E_{1\perp} = \sigma_f = \epsilon/\epsilon_0 \, \sigma$$

$$\Rightarrow \quad \epsilon_0 E_{2\perp} - \frac{\epsilon_0}{\epsilon} E_{1\perp} = \sigma$$

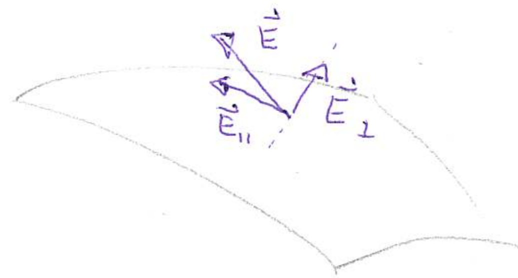
General boundary conditions

Generally sheets of surface charges and surface currents will produce jumps in $\vec{E}, \vec{D}, \vec{B}$ or $\vec{H}$.



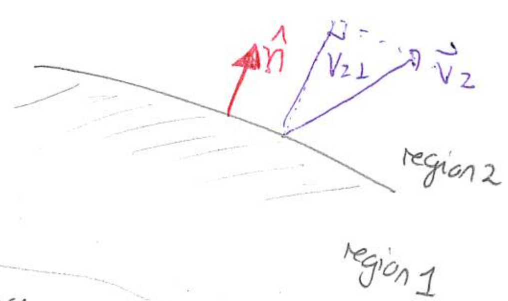
The jumps are usually described in terms of discontinuities in either

* perpendicular or
* tangential components



of fields. These can be constructed from the fields and the appropriate normal vector. Consider two regions as illustrated. Let

$\hat{n}$ be a normal vector from region 1 to region 2



Consider a vector field, $\vec{V}$. Let

$\vec{V}_2$ = be the field immediately above the surface in region 2

$\vec{V}_1$ = " " below the surface in region 1

Then the perpendicular component of $\vec{V}_2$ is

$$V_2^{\perp} = \vec{V}_2 \cdot \hat{n}$$

and the perpendicular component of $\vec{V}_1$ is

$$V_1^{\perp} = \vec{V}_1 \cdot \hat{n}$$

The tangential component is a vector

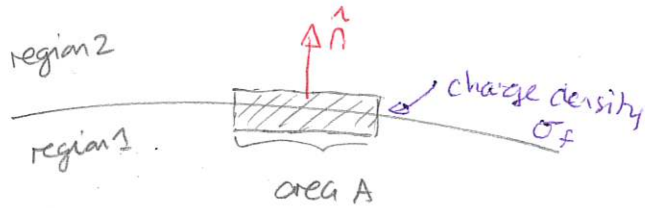
$$\vec{V}_{\parallel} = \vec{V} - \underbrace{(\vec{V}\cdot\hat{n})\hat{n}}_{\text{perpendicular component}}$$

Discontinuities in electric displacement

The electric displacement satisfies

$$\vec{\nabla}\cdot\vec{D} = \rho_{free}$$

$$\Rightarrow \quad \oint \vec{D}\cdot d\vec{a} = Q_{free\ enc.}$$



We apply this to a pillbox with infinitesimal area A and sides perpendicular to the surface. Then

$$\oint \vec{D}\cdot d\vec{a} = \underbrace{\cancel{\int \vec{D}\cdot d\vec{a}}}_{side} + \underbrace{\int \vec{D}\cdot d\vec{a}}_{top} + \underbrace{\int \vec{D}\cdot d\vec{a}}_{bottom} = \sigma_f A$$

$$\Rightarrow \quad \vec{D}_2\cdot\hat{n}\,A - \vec{D}_1\cdot\hat{n}\,A = \sigma_f A \quad \Rightarrow \quad D_2\cdot\hat{n} - D_1\cdot\hat{n} = \sigma_f$$

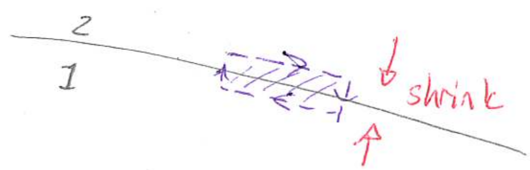
Thus we get equivalent statements:

$$\boxed{\begin{array}{l} \text{Let } \hat{n} \text{ be the normal from 1 to 2. Then.} \\ (\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = \sigma_f \quad \text{OR} \quad D_2^{\perp} - D_1^{\perp} = \sigma_f \end{array}}$$

Tangential component of $\vec{E}$

For the tangential component, consider

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$



Use the indicated Ampèrian loop.

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

Then shrinking the loop gives

$$\underbrace{\int \vec{E}\cdot d\vec{l}}_{\substack{left \\ \to 0}} + \underbrace{\int \vec{E}\cdot d\vec{l}}_{top} + \underbrace{\int \vec{E}\cdot d\vec{l}}_{\substack{right \\ \to 0}} + \underbrace{\int \vec{E}\cdot d\vec{l}}_{bottom} = -\frac{d}{dt}\underbrace{\int \vec{B}\cdot d\vec{a}}_{\to 0}$$

$$\Rightarrow \quad \underbrace{\int \vec{E}\cdot d\vec{l}}_{top} + \underbrace{\int \vec{E}\cdot d\vec{l}}_{bottom} = 0 \quad \Rightarrow \quad E_2^{\parallel} - E_1^{\parallel} = 0 \quad \Rightarrow \quad E_2^{\parallel} = E_1^{\parallel}$$

We can do this for any loop oriented like this: The result is

$$\vec{E}_2^{\parallel} = \vec{E}_1^{\parallel}$$

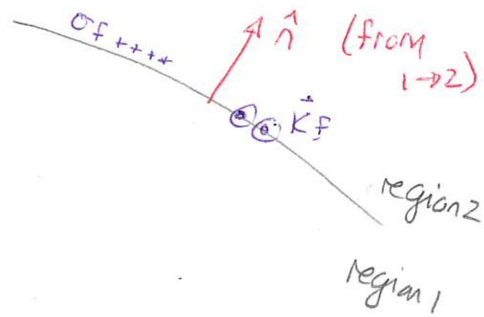
Continuing through Maxwells equations gives

$$D_2^{\perp} - D_1^{\perp} = \sigma_f \iff (\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = \sigma_f$$

$$\vec{E}_2^{\parallel} = \vec{E}_1^{\parallel} \iff (\vec{E}_2 - \vec{E}_1)\times\hat{n} = 0$$

$$B_2^{\perp} = B_1^{\perp} \iff (\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0$$

$$\vec{H}_2^{\parallel} - \vec{H}_1^{\parallel} = \vec{K}_f \times \hat{n} \iff \hat{n}\times(\vec{H}_2 - \vec{H}_1) = \vec{K}_f$$



Proof: 1) Electric displacement perpendicular

2) Magnetic field perpendicular (similar to $D^{\perp}$)

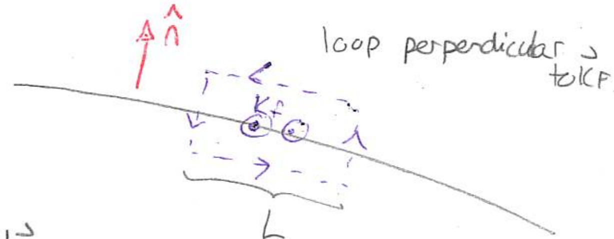
3) Electric field tangential - to convert into the cross product statement,

$$\hat{n}\times\left[(\vec{E}_2 - \vec{E}_1)\times\hat{n}\right] = (\vec{E}_2 - \vec{E}_1)\,\cancel{\hat{n}\hat{n}} - \hat{n}\left[(\vec{E}_2 - \vec{E}_1)\cdot\hat{n}\right]$$

$$= \left[\vec{E}_2 - (\vec{E}_2\cdot\hat{n})\,\hat{n}\right] - \left[\vec{E}_1 - (\vec{E}_1\cdot\hat{n})\,\hat{n}\right]$$

$$0 = \vec{E}_2^{\parallel} - \vec{E}_1^{\parallel}$$

4) $\vec{H}$ tangential field



$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

$$\Rightarrow \int_{\text{surface}} (\vec{\nabla}\times\vec{H})\cdot d\vec{a} = \int \vec{J}_f\cdot d\vec{a} + \frac{\partial}{\partial t}\int \vec{D}\cdot d\vec{a}$$

$$\Rightarrow \oint_{\text{loop}} \vec{H}\cdot d\vec{l} = K_f L + \underbrace{\frac{\partial}{\partial t}\int \vec{D}\cdot d\vec{a}}_{\text{goes to zero as loop shrinks.}}$$

$$\Rightarrow \quad \int_{\text{top}} \vec{H}\cdot d\vec{l} + \int_{\text{bottom}} \vec{H}\cdot d\vec{l} = K_f L.$$

Now let $\hat{k}$ be a unit vector along $\vec{K}_f$. On the top $d\vec{l} = \hat{k}\times\hat{n}\, dl$

$$\Rightarrow \int_{\text{top}} \vec{H}\cdot d\vec{l} = \vec{H}_2\cdot(\hat{k}\times\hat{n})L$$

Likewise on the bottom $d\vec{l} = -\hat{k}\times\hat{n}$

$$\Rightarrow \int_{\text{bottom}} \vec{H}\cdot d\vec{l} = -\hat{H}_1\cdot(\hat{k}\times\hat{n})L$$

Thus $\quad (\vec{H}_2 - \vec{H}_1)\cdot(\hat{k}\times\hat{n}) = K_f$

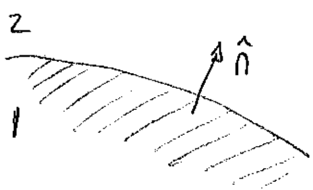
$$\Rightarrow \quad \hat{n}\cdot\left[(\vec{H}_2-\vec{H}_1)\times\hat{k}\right] = K_f$$

$$\Rightarrow \quad \hat{k}\cdot\left[\hat{n}\times(\vec{H}_2-\vec{H}_1)\right] = K_f \quad = \vec{K}_f\cdot\hat{k}$$

$$\Rightarrow \quad \hat{n}\times(\vec{H}_2-\vec{H}_1) = \vec{K}_f \quad \blacksquare$$

<u>Boundary conditions for Maxwell's equations.</u>

Similar reasoning gives



$$(\vec{E}_2-\vec{E}_1)\cdot\hat{n} = \sigma/\epsilon_0 \quad \Leftrightarrow \quad E_2^\perp - E_1^\perp = \sigma/\epsilon_0$$

$$(\vec{E}_2-\vec{E}_1)\times\hat{n} = 0 \quad \Leftrightarrow \quad \vec{E}_2^\parallel = \vec{E}_1^\parallel$$

$$(\vec{B}_2-\vec{B}_1)\cdot\hat{n} = 0 \quad \Leftrightarrow \quad B_2^\perp = B_1^\perp$$

$$\hat{n}\times(\vec{B}_2-\vec{B}_1) = \mu_0\vec{K} \quad \Leftrightarrow \quad \vec{B}_2^\parallel - \vec{B}_1^\parallel = \mu_0\vec{K}\times\hat{n}$$

Examples Infinite sheet of charge

$+\sigma$ $\quad \vec{E}_2 = \frac{\sigma}{2\epsilon_0}\hat{z}$ $\uparrow$

$\vec{E}_1 = -\frac{\sigma}{2\epsilon_0}\hat{z}$

$\vec{E}_2^{\parallel} = \vec{E}_1^{\parallel}$

$E_2^{\perp} - E_1^{\perp} = \frac{\sigma}{2\epsilon_0} - \left(\frac{-\sigma}{2\epsilon_0}\right) = \sigma/\epsilon_0$

Infinite sheet of current

$\uparrow\hat{n}$ $\quad \odot \ \vec{B}_2 = \frac{\mu_0 K}{2}\hat{x}$

$\longrightarrow \vec{K}$

$\otimes \ \vec{B}_1 = -\frac{\mu_0 K}{2}\hat{x}$

$\vec{B}_2^{\parallel} - \vec{B}_1^{\parallel} = \mu_0 K \hat{x}$

$= \mu_0 \vec{K} \times \hat{n}$

$B_2^{\perp} = B_1^{\perp}$

For potentials:

$$V_2 = V_1$$
$$(\vec{\nabla} V_2 - \vec{\nabla} V_1) \cdot \hat{n} = -\sigma/\epsilon_0$$
$$(\vec{\nabla} V_2 - \vec{\nabla} V_1) \times \hat{n} = 0$$

$$\vec{A}_2 = \vec{A}_1$$
$$\frac{\partial \vec{A}_2}{\partial n} - \frac{\partial \vec{A}_1}{\partial n} = -\mu_0 \vec{K}$$