Fri: HW by 5pm

Tues: Read 7.3.5
7.3.6

Magnetization and bound currents

The magnetic properties of a material can be described in terms of a distribution of point dipoles. The distribution is quantified by the magnetization $\vec{M}(\vec{r}\,')$, which is the dipole moment per unit volume. This then gives the scheme:

Known magnetization, $\vec{M}$

Determine vector potential

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

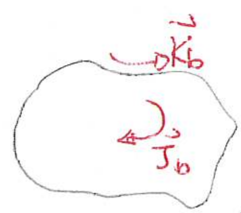
Magnetic field

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Bound currents

surface $\vec{K}_b = \vec{M} \times \hat{n}$

volume $\vec{J}_b = \vec{\nabla} \times \vec{M}$



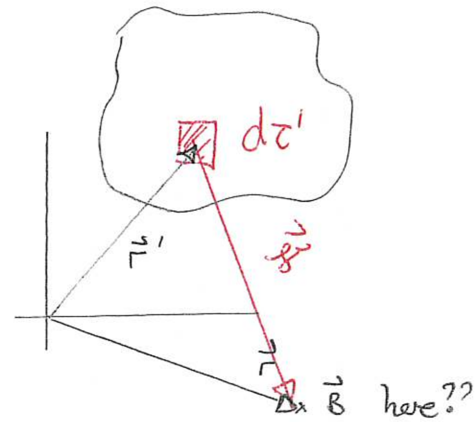
Use current densities to determine field

- Biot-Savart Law
- Ampère's Law

Give same result for $\vec{B}$

Recall that the computational tools, adapted for bound currents are as follows. The Biot-Savart Law

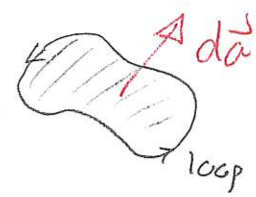
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\text{volume}} \frac{\vec{J}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' + \frac{\mu_0}{4\pi} \int_{\text{surface}} \frac{\vec{K}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} da'$$



The more convenient technique involves Ampère's Law.

For any closed loop

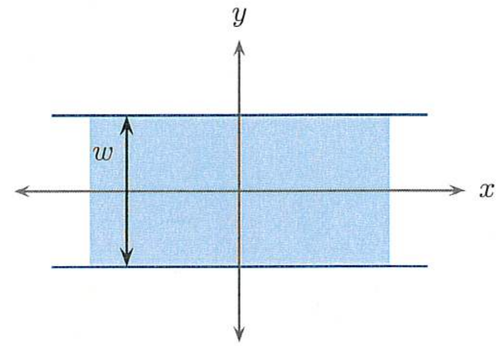
$$\oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$



where the enclosed current is

$$I_{enc} = \int_{\text{loop surface}} \vec{J}_b \cdot d\vec{a}$$

1 Uniformly magnetized slab



A slab of material extends infinitely along the z and x axes and has width w along the y axis. The origin of the axes is centered as illustrated. The material is magnetized in such a way that $\mathbf{M} = M_0\hat{\mathbf{z}}$ where $M_0 > 0$.

a) Determine expressions for the bound current densities and sketch these qualitatively.

b) In order to determine the magnetic field, we need to determine its direction. In general

$$\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}.$$

Use the Biot-Savart law to eliminate one component and the "current-reversal" argument to eliminate another.

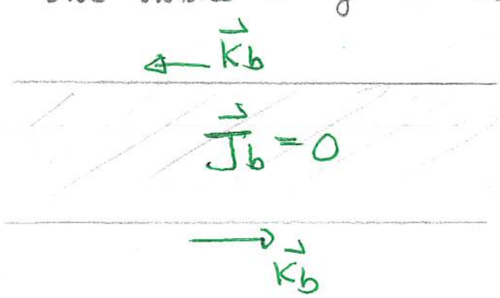
c) Choose an appropriate Ampèrian loop and apply Ampère's law to determine the magnetic field at all points.

Answer: a) $\vec{K}_b = \vec{M} \times \hat{n}$ and $\vec{J}_b = \vec{\nabla} \times \vec{M}$. Since $\vec{M}$ is constant inside,

$\vec{J}_b = 0$.

On the upper surface $\hat{n} = \hat{y} \Rightarrow \vec{K}_b = M_0 \hat{z} \times \hat{y} = -M_0\hat{x}$

On the lower surface $\hat{n} = -\hat{y} \Rightarrow \vec{K}_b = -M_0 \hat{z} \times \hat{y} = M_0\hat{x}$

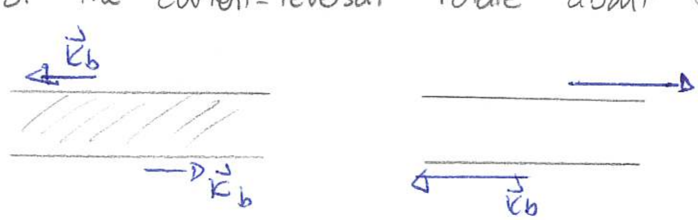


b) The Biot-Savart law contains $\vec{K} \times \hat{r}$ and this means that the magnetic field has no component along $\vec{K} \Rightarrow$ no $\hat{x}$ component.

Thus:

$$\vec{B} = B_y\hat{y} + B_z\hat{z}$$

For the current-reversal rotate about the y axis through $180°$



$\Rightarrow$ current reverses, field reverses, any B_y is unchanged $\Rightarrow B_y = 0$

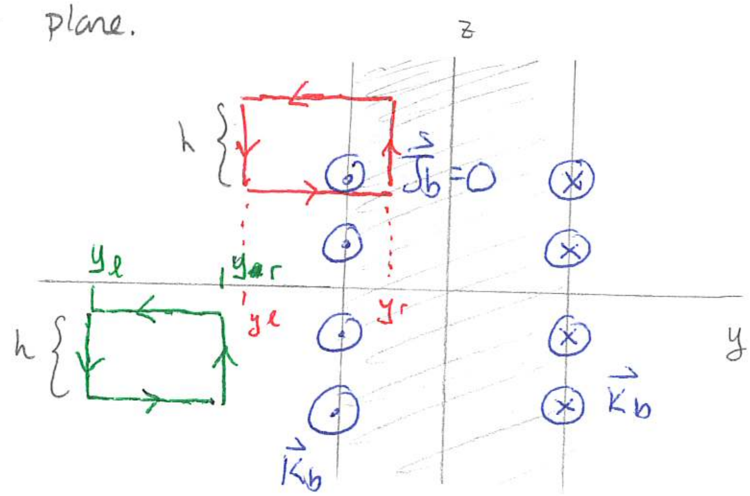
Thus

$$\vec{B} = B_z(y)\,\hat{z}$$

c) We need to:

* show field outside slab is uniform
* determine value of field outside slab
* determine value of field inside slab

To do this view in the yz plane.



Field outside the slab

We use Ampères Law with the green rectangle. Here

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 I_{enc}$$

But $I_{enc} = 0$ and this gives

$$\oint \vec{B}\cdot\vec{dl} = 0$$

$$\Rightarrow \int_{top} \vec{B}\cdot\vec{dl} + \int_{left} \vec{B}\cdot\vec{dl} + \int_{bottom} \vec{B}\cdot\vec{dl} + \int_{right} \vec{B}\cdot\vec{dl} = 0$$

Along the top and bottom $\vec{dl} = \pm dy\,\hat{y}$ and $\vec{B}\cdot\vec{dl} = 0$. So these integrals give zero.

Thus

$$\int_{left} \vec{B}\cdot\vec{dl} + \int_{right} \vec{B}\cdot\vec{dl} = 0$$

Then on the right $\vec{B} = B_z(y_r)\,\hat{z}$ and $\vec{dl} = dz\,\hat{z} \Rightarrow \vec{B}\cdot\vec{dl} = B_z(y_r)\,dz$

$$\Rightarrow \int_{right} \vec{B}\cdot\vec{dl} = \int B_z(y_r)\,dz = h\,B_z(y_r)$$

On the left $\vec{B} = B_z(y_l)\,\hat{z}$ and $\vec{dl} = -dz\,\hat{z} \Rightarrow \vec{B}\cdot\vec{dl} = -B_z(y_l)\,dz$

$$\Rightarrow \int_{left} \vec{B}\cdot\vec{dl} = -h\,B_z(y_l)$$

Thus

$$h\left[B_z(y_r) - B_z(y_\ell)\right] = 0 \quad \Rightarrow \quad B_z(y_r) = B_z(y_\ell)$$

This is true for all y_r, y_ℓ left of the slab. So B is uniform here. Then as $y \to -\infty$ the two sheets of charge will appear to overlap and this gives an apparent current density of zero $\Rightarrow$ $B \to 0$ as $y \to -\infty$. The same arguments work on the right. So

$$\vec{B} = 0 \quad \text{outside the slab.}$$

Field inside the slab

Use the red Gaussian surface. Then

$$\oint \vec{B} \cdot \vec{d\ell} = \mu_0 I_{enc}$$

$$\Rightarrow \quad \underbrace{\cancel{\int_{top} \vec{B} \cdot \vec{d\ell}}}_{0 \;\; \vec{B} \perp \vec{d\ell}} + \underbrace{\cancel{\int_{left} \vec{B} \cdot \vec{d\ell}}}_{0 \;\; \vec{B} = 0} + \underbrace{\cancel{\int_{bottom} \vec{B} \cdot \vec{d\ell}}}_{0 \;\; \vec{B} \perp \vec{d\ell}} + \int_{right} \vec{B} \cdot \vec{d\ell} = \mu_0 K_b h.$$

$$\Rightarrow \quad B_z(y_r) \cancel{h} = \mu_0 K_b \cancel{h} \quad \Rightarrow \quad B_z(y_r) = \mu_0 K_b = \mu_0 M_0$$

This is also independent of y_r.

Entire field

Summarizing

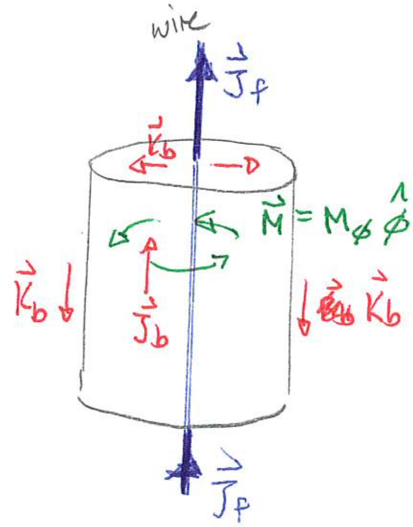
$$\boxed{\vec{B} = \begin{cases} \mu_0 M \hat{z} & \text{inside slab} \\ 0 & \text{outside slab.} \end{cases}}$$

Note that $\vec{B} = \mu_0 \vec{M}$

Auxiliary magnetic field

In the example we saw that $\vec{B} = \mu_0 \vec{M}$. This will be true for any appropriately symmetric magnetization $\vec{M} = M(y)\,\hat{z}$. Could we arrive at this without all the machinery of Ampère's Law?

Consider a magnetized material in the presence of a free current. The magnetic field generated by free and bound currents satisfies



$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

where

$$\vec{J} = \vec{J}_{free} + \vec{J}_b$$

is the total current density. Then

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

gives:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{free} + \mu_0 \vec{\nabla} \times \vec{M}$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{B} - \mu_0 \vec{\nabla} \times \vec{M} = \mu_0 \vec{J}_f$$

$$\Rightarrow \quad \vec{\nabla} \times \left[\vec{B} - \mu_0 \vec{M}\right] = \mu_0 \vec{J}_f \qquad \Rightarrow \qquad \vec{\nabla} \times \left[\frac{1}{\mu_0}\vec{B} - \vec{M}\right] = \vec{J}_f$$

We define the auxiliary magnetic field as

$$\boxed{\vec{H} := \frac{1}{\mu_0}\vec{B} - \vec{M}}$$

Then

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_f}$$

Then

$$\vec{\nabla}\cdot\vec{H} = \cancel{\frac{1}{\mu_0}\vec{\nabla}\cdot\vec{B}}_{\,0} - \vec{\nabla}\cdot\vec{M}$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$$

Thus

> Given free current density $\vec{J}_f$ and magnetization $\vec{M}$ the auxiliary magnetic field satisfies
>
> $$\vec{\nabla}\times\vec{H} = \vec{J}_f$$
> $$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$$

We can adapt some rules from magnetostatics to determine the auxiliary magnetic field. Stoke's theorem will give

> For any closed loop
>
> $$\oint \vec{H}\cdot d\vec{l} = I_{\text{free enc}}$$
>
> where the enclosed free current is
>
> $$I_{\text{free enc}} = \int_{\text{any surface banded by loop.}} \vec{J}_{\text{free}}\cdot d\vec{a}$$
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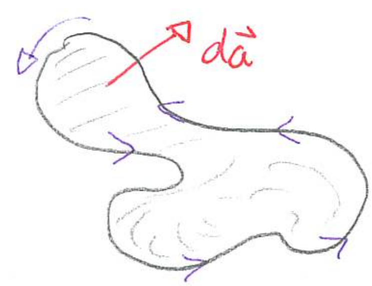
> 
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This means we can determine fields by:

Known free current $\vec{J}_F$ $\rightarrow$ Determine $\vec{H}$ $\rightarrow$ Then magnetic field is $\vec{B} = \mu_0(\vec{H}+\vec{M})$

Known magnetization $\vec{M}$ $\rightarrow$ Then magnetic field is $\vec{B} = \mu_0(\vec{H}+\vec{M})$

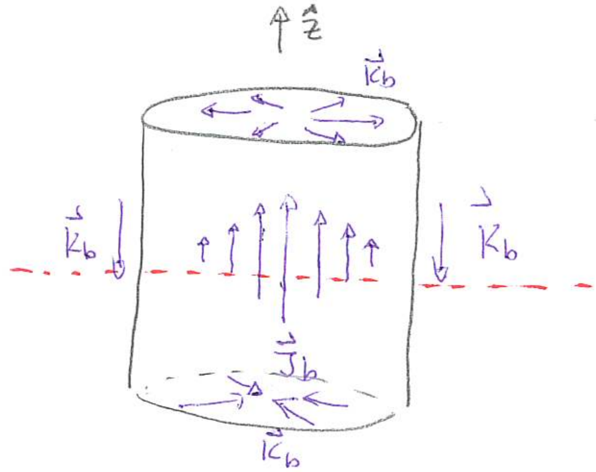
We could have used this in the example. Symmetries and the lack of free currents give $\vec{H}=0 \Rightarrow \vec{B}=\mu_0\vec{M}$.

2 Magnetic field in the presence of a cylindrical material

A cylindrical material with radius R carries magnetization $\mathbf{M} = M\hat{\boldsymbol{\phi}}$ where $M > 0$ is constant. There is no free current.

a) Suppose that the material has a finite length with ends that are perpendicular to the axis. Determine the direction of the bound volume and surface currents.

b) Determine the auxiliary field if the cylinder is infinite in length and use this to determine the magnetic field.



Answer: a) In the interior

$$\vec{J}_b = \vec{\nabla}\times\vec{M}$$

$$= \left[\frac{1}{s}\cancel{\frac{\partial M_z}{\partial\phi}} - \frac{\partial M_\phi}{\partial z}\right]\hat{s} + \left[\cancel{\frac{\partial M_s}{\partial z}} - \cancel{\frac{\partial M_z}{\partial s}}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sM_\phi) - \cancel{\frac{\partial M_s}{\partial \phi}}\right]\hat{z}$$

$$= -\cancel{\frac{\partial M_\phi}{\partial z}}^{\,0}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}(sM)\hat{z} \Rightarrow \vec{J}_b = \frac{M}{s}\hat{z}$$

On the top $\hat{n} = \hat{z}$ and $\vec{K}_b = \vec{M}\times\hat{n}$ gives

$$\vec{K}_b = M\hat{\phi}\times\hat{z} \Rightarrow \vec{K}_b = M\hat{s}$$

On the bottom $\hat{n} = -\hat{z}$ and $\vec{K}_b = \vec{M}\times\hat{n}$ gives

$$\vec{K}_b = -M\hat{\phi}\times\hat{z} \Rightarrow \vec{K}_b = -M\hat{s}$$

On the curved surface $\hat{n} = \hat{s}$

$$\Rightarrow \vec{K}_b = M\hat{\phi}\times\hat{s} = -M\hat{z}$$

b) In general $\vec{B} = B_s(s)\hat{s} + B_\phi(s)\hat{s} + B_z(s)\hat{s}$.

Since current flows along $\hat{z}$ we get $B_z(s) = 0$

$$\Rightarrow \vec{B} = B_s(s)\hat{s} + B_\phi(s)\hat{s}$$

Then a rotation about the red axis through 180' flips the current and would flip the field. However, any $\hat{s}$ component of $\vec{B}$ would be non-zero. So

$$\vec{B} = B_\phi(s)\,\hat{\phi}$$

Then

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$= \left[\frac{1}{\mu_0} B_\phi(s) - M\right]\hat{\phi} \equiv H_\phi(s)\,\hat{\phi}$$

We use an Amperian loop which is a circle of radius s in the x-y plane. Then

$$\oint \vec{H}\cdot d\vec{l} = I_{f\,enc}$$

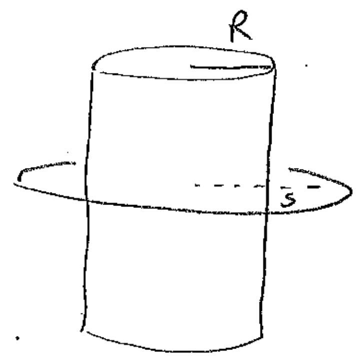


illustration for $s > R$.

In all cases $I_{f\,enc} = 0$.

For the integral

$$\left.\begin{array}{r} s' = s \\ z' = 0 \\ 0 \leq \phi' \leq 2\pi \end{array}\right\} \Rightarrow d\vec{l} = s'd\phi'\,\hat{\phi}$$
$$= s\,d\phi'\,\hat{\phi}$$

$$\Rightarrow \vec{H}\cdot d\vec{l} = s H_\phi(s)\, d\phi'$$

$$\Rightarrow \oint \vec{H}\cdot d\vec{l} = \int_0^{2\pi} d\phi'\, s H_\phi(s) = 2\pi s\, H_\phi(s) = 0$$

$$\Rightarrow H_\phi = 0$$

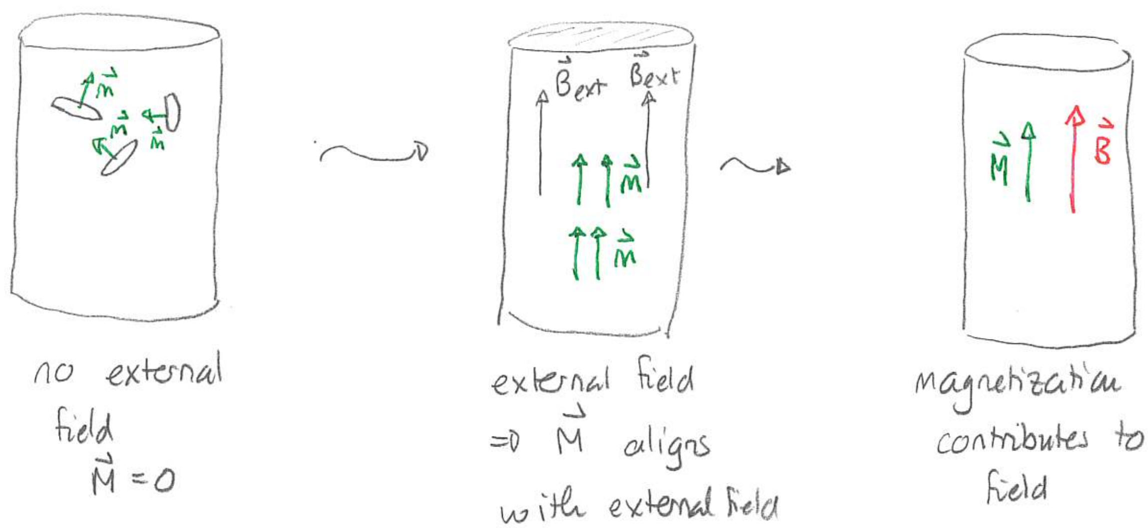
Thus $\vec{H} = 0$. Then

$$0 = \frac{1}{\mu_0}\vec{B} - \vec{M} \Rightarrow \vec{B} = +\mu_0\vec{M}$$

$$\Rightarrow \vec{B} = \begin{cases} \mu_0 M\,\hat{\phi} & s < R \\ 0 & s > R \end{cases}$$

Magnetic materials

We need to describe how magnetization is produced and could consider



Linear magnetic materials are a type of material for which

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is a unitless, material-dependent magnetic susceptibility. Then

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\Rightarrow \quad \mu_0 \vec{H} = \vec{B} - \mu_0 \chi_m \vec{H}$$

$$\Rightarrow \quad \vec{B} = \mu_0 (1+\chi_m)\vec{H}$$

We define the permeability of the material as

$$\boxed{\mu := \mu_0 (1+\chi_m)}$$

Then

$$\boxed{\vec{B} = \mu \vec{H}}$$

3 Magnetic field in the presence of a cylindrical material surrounding a current-carrying wire

An infinitely long cylindrical material with radius R surrounds an infinitely long wire that carries current I. The material is linear with permeability μ and is surrounded by free space.

a) Determine the auxiliary field at all locations.

b) Determine the magnetic field at all locations.

c) Determine the magnetization at all locations and determine the bound current densities.

Answer: a) We will use

$$\oint_{\text{loop}} \vec{H}\cdot d\vec{l} = I_{\text{free enc}}.$$

Then $\vec{H}$ is determined solely by $\vec{J}_{\text{free}}$ and $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\frac{1}{\mu}\vec{B}$

$$= \frac{1}{\mu}\vec{\nabla}\cdot\vec{B} = 0.$$

This means that $\vec{H}$ is determined solely by $\vec{J}_{\text{free}}$.

The usual symmetry arguments give

$$\vec{H} = H_\phi(s)\,\hat{\phi}$$

We use as an Amperian loop, a loop of radius s in the transverse plane. Then

$$\oint \vec{H}\cdot d\vec{l} = I_{\text{free enc}} \Rightarrow H_\phi(s)\,2\pi s = I$$

$$\Rightarrow \boxed{\vec{H} = \frac{I}{2\pi s}\hat{\phi}}$$

b) Inside the material

$$\vec{B} = \mu\vec{H} \Rightarrow \vec{B} = \frac{\mu I}{2\pi s}\hat{\phi}$$

Outside we replace $\mu \to \mu_0$. Thus

$$\boxed{\vec{B} = \begin{cases} \dfrac{\mu I}{2\pi s}\hat{\phi} & \text{inside} \\ \dfrac{\mu_0 I}{2\pi s}\hat{\phi} & \text{outside} \end{cases}}$$

c) We use, inside the material

$$\vec{M} = \chi_M \vec{H}$$

$$= \frac{\chi_M I}{2\pi s} \hat{\phi}$$

But $\mu = \mu_0(1+\chi_M) \Rightarrow \mu/\mu_0 - 1 = \chi_M$

$$\Rightarrow \quad \vec{M} = \left[\frac{\mu}{\mu_0} - 1\right] \frac{I}{2\pi s} \hat{\phi}$$

On the surface $\hat{n} = \hat{s}$ and the bound surface current density is

$$\vec{K}_b = \vec{M} \times \hat{n}$$

$$= \left(\frac{\mu}{\mu_0} - 1\right) \frac{I}{2\pi s} \underbrace{\hat{\phi} \times \hat{s}}_{-\hat{z}} \quad \Rightarrow \quad \vec{K}_b = -\left(\frac{\mu}{\mu_0} - 1\right) \frac{I}{2\pi R} \hat{z}$$

In the interior

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

$$= \chi_m \vec{\nabla} \times \vec{H}$$

$$= \chi_m \left\{ \cancel{\frac{1}{s} \frac{\partial}{\partial s}(s H_\phi)}^{0} \right\} \hat{z} \quad \Rightarrow \quad \vec{J}_b = 0$$