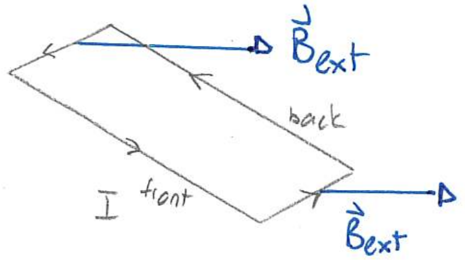
Tues: HW by 5pm

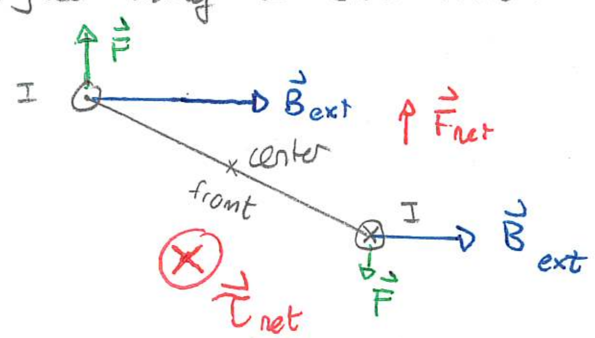
Thurs: Read

Fri: HW by 5pm

Magnetic dipole in an external field

The model of material as a collection of magnetic dipoles in an external field requires an analysis of the response of a dipole to an external field. A physical model is a rectangular current loop in a field. Consider the illustrated example where a loop is in a non-uniform field. This is best analyzed using a side view:





The forces on the front and back cancel. We see that

1) If the field is not uniform (has a gradient) then there will be a net force on the loop.
2) For the field illustrated, there will be a net torque on the loop about its center. This is as illustrated

We would like to determine the net force and net torque directly from the external field $\vec{B}_{ext}$ and the dipole moment, $\vec{m}$.

A general result is

If a dipole $\vec{m}$ is placed in an external magnetic field $\vec{B}$ then the force exerted on the dipole is

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

If the field is uniform then the net torque is

$$\vec{\tau} = \vec{m}\times\vec{B}$$

Proof: Force

In general the force on a current is

$$\vec{F} = \int_{\text{all space}} \vec{J}(\vec{r}\,')\times\vec{B}(\vec{r}\,')\,d\tau' \qquad -(1)$$

We will determine the force to first order in $\vec{J}$ and $\vec{B}$. The eventual result will involve the monopole moment

$$\int_{\text{all space}} \vec{J}(\vec{r}\,')\,d\tau' = 0 \qquad -(2)$$

and the dipole moment

$$\vec{m} = \frac{1}{2}\int_{\text{all space}} \vec{r}\,'\times\vec{J}(\vec{r}\,')\,d\tau' \qquad -(3)$$

The strategy is to expand $\vec{B}(\vec{r}\,')$ about a reference point $\vec{r}$. The expansion will yield terms of the form

$$\begin{pmatrix}\text{other}\\ \text{stuff}\end{pmatrix}\int_{\text{all space}} \vec{J}(\vec{r}\,')\times\vec{r}\,'$$

in the expression for force.

Standard calculus gives:

$$B_x(\vec{r}') = B_x(\vec{r}) + (\vec{r}'-\vec{r})\cdot \underbrace{\vec{\nabla} B_x}_{\frac{\partial B_x}{\partial x}\hat{x} + \frac{\partial B_x}{\partial y}\hat{y} + \frac{\partial B_x}{\partial z}\hat{z}} + \text{higher terms.}$$

derivatives w.r.t. unprimed evaluated at unprimed

This is extended componentwise to

$$\vec{B}(\vec{r}') = \vec{B}(\vec{r}) + \left[(\vec{r}'-\vec{r})\cdot\vec{\nabla}\right]\vec{B} + \cdots$$

Thus

$$\vec{F} = \int_{\text{all space}} \vec{J}(\vec{r}') \times \vec{B}(\vec{r})\, d\tau' + \int \vec{J}(\vec{r}') \times \left[(\vec{r}'\cdot\vec{\nabla})\vec{B}\right] d\tau' + \int \vec{J}(\vec{r}') \times \left[(\vec{r}\cdot\vec{\nabla})\vec{B}\right] d\tau' + \cdots$$

The red indicates primed and the blue unprimed. The first term is:

$$\left[\int \vec{J}(\vec{r}')\, d\tau'\right] \times B(\vec{r}) = 0 \times \vec{B}(\vec{r}) \quad \text{by eqn (2)}$$
$$= 0$$

Similarly the last term is zero. Thus we get

$$\vec{F} = \int \vec{J}(\vec{r}') \times \left[(\vec{r}'\cdot\vec{\nabla})\vec{B}\right] d\tau' + \text{smaller terms}$$

The remaining steps involve rearranging the integrand and using vector calculus identites. The integrand is

$$\vec{J}(\vec{r}') \times \left(x'\frac{\partial}{\partial x}\vec{B} + y'\frac{\partial}{\partial y}\vec{B} + z'\frac{\partial}{\partial z}\vec{B}\right) = x'\vec{J}(\vec{r}') \times \frac{\partial}{\partial x}\vec{B} + y'\vec{J}(\vec{r}') \times \frac{\partial}{\partial y}\vec{B} + z'\vec{J}(\vec{r}') \times \frac{\partial}{\partial z}\vec{B}$$

$$= \frac{\partial}{\partial x}\left[x'\vec{J}(\vec{r}') \times \vec{B}\right] + \frac{\partial}{\partial y}\left[y'\vec{J}(\vec{r}') \times \vec{B}\right] + \frac{\partial}{\partial z}\left[z'\vec{J}(\vec{r}') \times \vec{B}\right]$$

But $x' = \hat{x}\cdot\vec{r}\,'$
$y' = \hat{y}\cdot\vec{r}\,'$

gives

$$\vec{F} = \frac{\partial}{\partial x}\left[\int_{\text{all space}} (\hat{x}\cdot\vec{r}\,')\,\vec{J}(\vec{r}\,')\times\vec{B}\,d\tau'\right] + \frac{\partial}{\partial y}\left[\int_{\text{all space}} (\hat{y}\cdot\vec{r}\,')\,\vec{J}(\vec{r}\,')\times\vec{B}\,d\tau'\right] + \frac{\partial}{\partial z}\left[\ldots\right] \tag{4}$$

This is taking the form of a gradient but we need to manipulate each argument to eventually introduce the the dipole moment. Now we use an intermediate result. Consider any localized current density $\vec{J}(\vec{r}\,')$ that is not time varying. Then $\vec{\nabla}'\cdot\vec{J}(\vec{r}\,') = 0$ by the continuity equation.

We consider

$$\int_{\text{all space}} (\vec{r}\cdot\vec{r}\,')\,\vec{J}(\vec{r}\,')\,d\tau'$$

and we will show

$$\int (\vec{r}\cdot\vec{r}\,')\,\vec{J}(\vec{r}\,')\,d\tau' = -\tfrac{1}{2}\,\vec{r}\times\left(\int \vec{r}\,'\times\vec{J}(\vec{r}\,')\,d\tau'\right) \qquad -(5)$$

$$= -\vec{r}\times\vec{m} \qquad -(6)$$

To show (5) consider

$$\vec{r}\times(\vec{r}\,'\times\vec{J}(\vec{r}\,')) = \vec{r}\,'(\vec{r}\cdot\vec{J}(\vec{r}\,')) - \vec{J}(\vec{r}\,')(\vec{r}\cdot\vec{r}\,')$$

Thus

$$\vec{r}\times\int_{\text{all space}} \vec{r}\,'\times\vec{J}(\vec{r}\,')\,d\tau' = \int_{\text{all space}} \vec{r}\,'[\vec{r}\cdot\vec{J}(\vec{r}\,')]\,d\tau' - \int_{\text{all space}} (\vec{r}\cdot\vec{r}\,')\,\vec{J}(\vec{r}\,')\,d\tau'$$

Now we can show

$$\int \vec{r}' [\vec{r}\cdot\vec{J}(\vec{r}')]\, d\tau' = -\int (\vec{r}\cdot\vec{r}')\, \vec{J}(\vec{r}')\, d\tau' \qquad -(7)$$

Then, this would give

$$\vec{r}\times \int_{\text{all space}} \vec{r}'\times\vec{J}(\vec{r}')\, d\tau' = -2\int_{\text{all space.}} (\vec{r}\cdot\vec{r}')\, \vec{J}(\vec{r}')\, d\tau'$$

and that gives (5). So we have to prove (7). To do this: Consider any vector $\vec{S}$.

$$\int (\vec{S}\cdot\vec{r}')\, [\vec{r}\cdot\vec{J}(\vec{r}')]\, d\tau' = \int (\vec{S}\cdot\vec{r}')\, [\vec{\nabla}'(\vec{r}'\cdot\vec{r})\cdot\vec{J}(r')]\, d\tau'$$

$$= \int_{\text{all space}} \vec{\nabla}'\cdot[(\vec{S}\cdot\vec{r}')(\vec{r}\cdot\vec{r}')\vec{J}(\vec{r}')]\, d\tau'$$

By the divergence theorem this $\to 0$ for localized charges.

$$-\vec{S}\cdot\int \hat{\imath}'(\vec{r}\cdot\vec{r}')\vec{J}(\vec{r}')\, d\tau'$$

$$-\int (\vec{S}\cdot\vec{r}')(\vec{r}\cdot\vec{r}')\underbrace{\vec{\nabla}'\cdot\vec{J}(\vec{r}')}_{=0}\, d\tau'$$

Thus

$$\vec{S}\cdot\left[\int_{\text{all space}} \vec{r}'(\vec{r}\cdot\vec{J}(\vec{r}'))\, d\tau'\right] = -\vec{S}\cdot\left[\int_{\text{all space}} \hat{\imath}'(\vec{r}\cdot\vec{r}')\vec{J}(\vec{r}')\, d\tau'\right]$$

This is true for all $\vec{S}$ and that proves (7) and therefore (6)

Thus with (4) and (6)

$$\vec{F} = -\left\{\frac{\partial}{\partial x}\left[(\hat{x}\times\vec{m})\times\vec{B}\right] + \frac{\partial}{\partial y}\left[(\hat{y}\times\vec{m})\times\vec{B}\right] + \frac{\partial}{\partial z}[\ldots]\right.$$

$$= +\frac{\partial}{\partial x}\left[\vec{B}\times(\hat{x}\times\vec{m})\right] + \ldots$$

$$= +\frac{\partial}{\partial x}\left[\hat{x}(\vec{B}\cdot\vec{m})\right] - \frac{\partial}{\partial x}\left[\vec{m}(\vec{B}\cdot\hat{x})\right] + \ldots$$

$$= +\frac{\partial}{\partial x}(\vec{B}\cdot\vec{m})\hat{x} - \vec{m}\frac{\partial B_x}{\partial x} + \frac{\partial}{\partial y}(\vec{B}\cdot\vec{m})\hat{y} - \vec{m}\frac{\partial B_y}{\partial y} + \ldots$$

Thus

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) - \vec{m}\,\underbrace{(\vec{\nabla}\cdot\vec{B})}_{\Rightarrow = 0}$$

$$\Rightarrow \quad \vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

Torque : See Griffiths Ch 6.1

1 Magnetic dipole in a field with a gradient

Suppose that $\mathbf{B} = B(z)\hat{\mathbf{z}}$ where $B(z)$ is the field magnitude which depends on z only. A magnetic dipole is propelled along the $\hat{\mathbf{x}}$ direction and then enters this field. Determine the force exerted on this particle and describe how the process of observing the trajectory of the particle as it traverses the field can be used to determine a component of its dipole moment. This is the basic idea behind the famous Stern-Gerlach experiment.

Answer:

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

$$= \vec{\nabla}\left[\vec{m}\cdot B(z)\hat{z}\right] = \vec{\nabla}\left[B(z)\, m_z\right]$$

$$= \cancel{\frac{\partial}{\partial x}(B(z)m_z)}\hat{x} + \frac{\partial}{\partial y}\cancel{(B(z)m_z)}\hat{y} + \frac{\partial}{\partial z}(B(z)m_z)\hat{z}$$

$$\vec{F} = \frac{\partial B}{\partial z} m_z \hat{z}$$

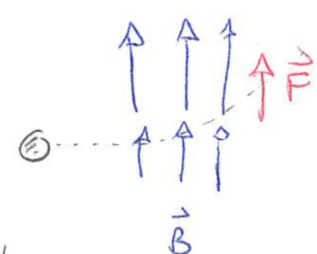
The force will be

- along the z axis
- proportional to the z-component of $\vec{m}$

Suppose $\frac{\partial B}{\partial z} > 0$. Then.

$m_z > 0 \Rightarrow$ deflected up

$m_z < 0 \Rightarrow$ deflected down

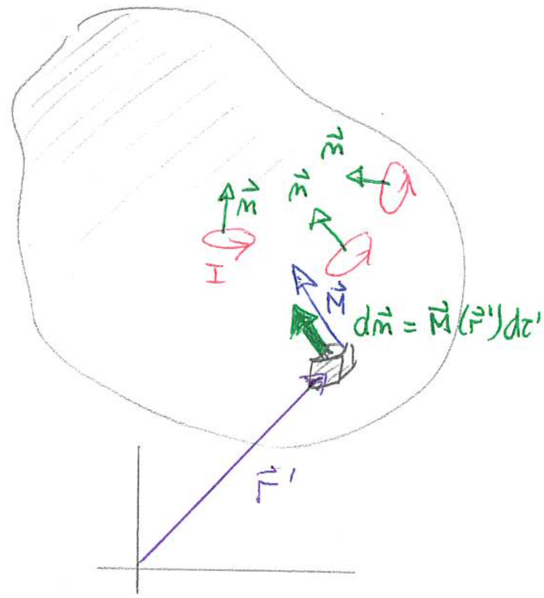


Measuring the deflection will eventually give m_z

DEMO: Toutquantique SG

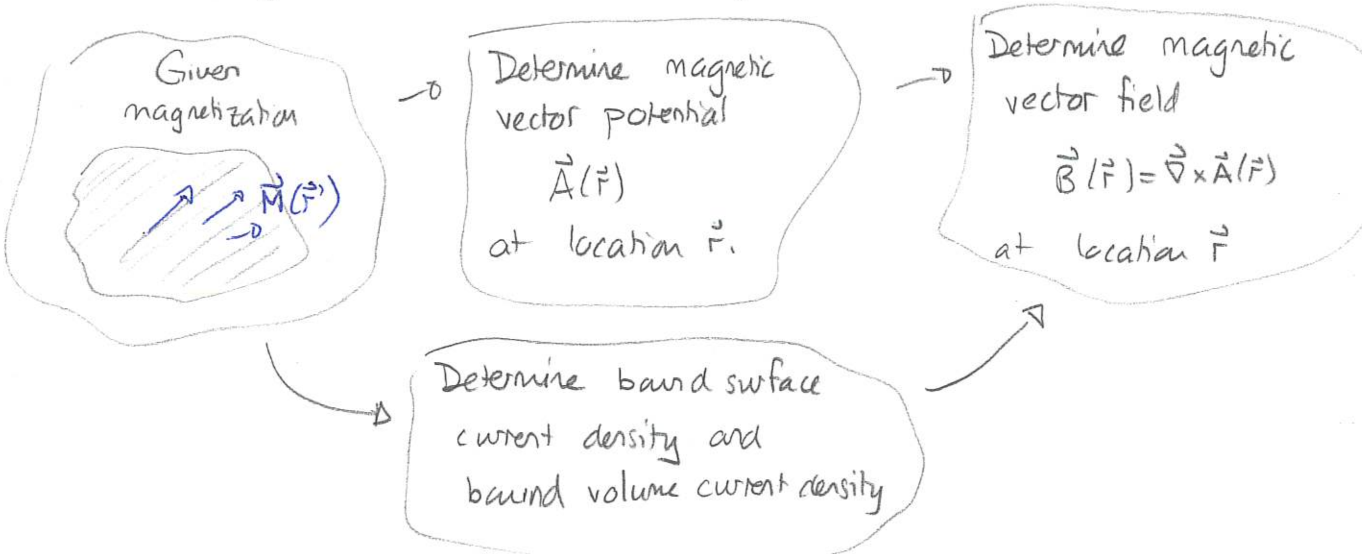
Magnetization

Consider a material which can be modeled as a collection of magnetic dipoles. A simple example would be as illustrated. We could, in this case, describe the material via



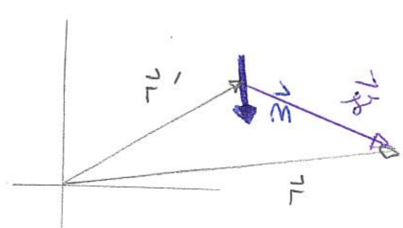
> The magnetization of the material $\vec{M}(\vec{r}\,')$ is a vector such that the magnetic dipole moment in a region of volume $d\tau'$ at $\vec{r}\,'$ is
>
> $$d\vec{m} = \vec{M}(\vec{r}\,')d\tau'$$
>
> units A/m

This then aligns with the standard magnetostatic method to calculate fields:



The core of this is that, for a point magnetic dipole, $\vec{m}$ at $\vec{r}\,'$ the vector potential is

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$



Then over an extended distribution the small contributions are

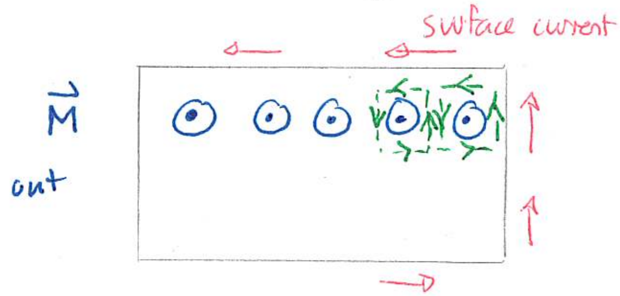
$$d\vec{A} = \frac{\mu_0}{4\pi}\frac{d\vec{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\frac{\vec{M}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

Thus, given magnetization $\vec{M}(\vec{r}\,')$ the vector potential is:

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int\limits_{\substack{\text{all current}\\ \text{distribution}}} \frac{\vec{M}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'}$$

This would offer a direct method for calculating the potential. We expect that an equivalent method would be via bound currents. Consider two situations:

Uniform magnetization



* It appears that currents in the interior cancel and that there will be a surface current, described by some bound surface current density

$$\vec{K}_b(\vec{r}\,')$$

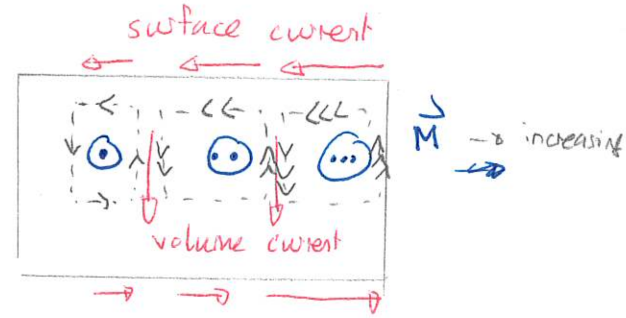
* The bound surface current density is perpendicular to both $\vec{M}(\vec{r}\,')$ and the normal vector to the surface.

* The vector potential produced by a surface current is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

and the bound surface current should yield the same potential as from $\vec{M}$

Gradient magnetization



* The currents in the interior will not cancel. There should be a bound volume current density

$$\vec{J}_b(\vec{r}\,')$$

* The bound volume current density will be related to a gradient in $\vec{M}$

* The vector potential produced by the volume current is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

The bound volume current density (and associated surface current density) should give the same potential as from $\vec{M}$

We can prove that:

Given a magnetization $\vec{M}(\vec{r}\,')$ the magnetic vector potential is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_b(\vec{r}\,')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\int \frac{\vec{K}_b(\vec{r}\,')}{\mathscr{r}}\, da'$$

where $\vec{\mathscr{r}}, \vec{r}, \vec{r}\,'$ are defined as usual and the bound volume current density is:

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

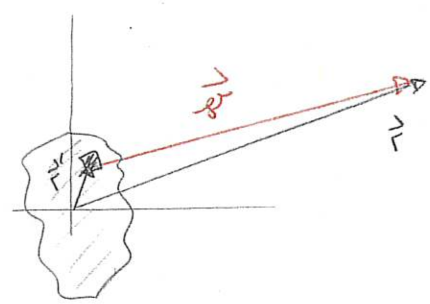
and the bound surface current density is

$$\vec{K}_b = \vec{M} \times \hat{n}$$

where $\hat{n}$ is the normal to the surface

Proof: Start with

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$



We first replace $\hat{\mathscr{r}}/\mathscr{r}^2$ using

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)$$

where $\vec{\nabla}' = \hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}$

$$\mathscr{r} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

Thus:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \vec{M}(\vec{r}') \times \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

We can then integrate by parts (in an effort to switch the $\vec{\nabla}'$ operator).

$$\vec{\nabla} \times (f\vec{C}) = f(\vec{\nabla} \times \vec{C}) - \vec{C} \times \vec{\nabla} f$$

$$\Rightarrow \quad \vec{C} \times \vec{\nabla} f = f(\vec{\nabla} \times \vec{C}) - \vec{\nabla} \times (f\vec{C}).$$

Thus:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}} \vec{\nabla}' \times \vec{M}(\vec{r}')\, d\tau' - \frac{\mu_0}{4\pi} \int \vec{\nabla}' \times \left(\frac{1}{\mathscr{r}} \vec{M}(\vec{r}')\right) d\tau'$$

We can restrict the integrals to regions where the magnetization is non-zero. Then:

$$\underbrace{\int (\vec{\nabla} \times \vec{C})\, d\tau'}_{\text{volume}} = - \underbrace{\oint \vec{C} \times d\vec{a}}_{\text{surface}}$$

gives:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}_b(\vec{r}')}{\mathscr{r}} d\tau' + \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}} \underbrace{\vec{M}(\vec{r}') \times d\vec{a}'}_{\vec{M}(\vec{r}') \times \hat{n}\, da'}$$

where $\vec{J}_b = \vec{\nabla} \times M(\vec{r}')$. Then with

$\vec{K}_b = \vec{M}(\vec{r}') \times \hat{n}$

we get the result. ☒

So we now have

Given magnetization $\vec{M}(\vec{r}')$

$\rightarrow$

Determine bound current densities:

surface $\vec{K}_b = \vec{M} \times \hat{n}$

volume $\vec{J}_b = \vec{\nabla} \times \vec{M}$

$\swarrow$

Use techniques from magnetostatics (Biot-Savart Law, Ampère's Law) to calculate magnetic fields from bound currents

2 Magnetized cylinder

Consider a cylinder with radius R and axis along $\hat{\mathbf{z}}$. Using a cylindrical coordinate system with origin at the center of the cylinder, the magnetization is

$$\mathbf{M}(\mathbf{r}') = M(s)\hat{\mathbf{z}}$$

where $M(s) > 0$ has units of A/m.

a) Suppose that $M(s) = M_0 > 0$ which is independent of radial distance. Sketch a collection of dipoles that could produce this. Use your sketch to predict whether the bound surface and volume current densities are zero, and if not what their directions are.

b) Determine expressions of the bound surface and volume current densities. Do they agree with your answer to the previous part?

c) Suppose that $M(s) = \alpha s$ where $\alpha > 0$ has units of A/m^2. Sketch a collection of dipoles that could produce this. Use your sketch to predict whether the bound surface and volume current densities are zero, and if not what their directions are.

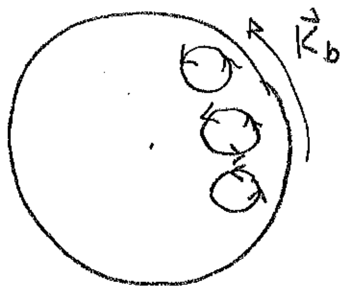
d) Determine expressions of the bound surface and volume current densities. Do they agree with your answer to the previous part?

Suppose that the magnetization is

$$\mathbf{M}(\mathbf{r}') = M_0 \cos\phi \hat{\mathbf{z}}$$

e) Determine expressions of the bound surface and volume current densities. Do they agree with what a sketch would predict?

Answer: a)



bound volume currents $\rightarrow 0$

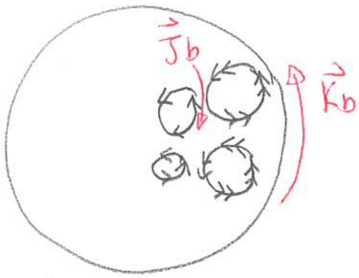
" surface currents $\rightarrow \hat{\phi}$ direction

b) $\vec{K}_b = \hat{M} \times \hat{n} = M_0 \underbrace{\hat{z} \times \hat{s}}_{\hat{\phi}} \Rightarrow \boxed{\vec{K}_b = M_0 \hat{\phi}}$ agrees!

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \vec{\nabla} \times (M_0 \hat{z}) \Rightarrow \vec{J}_b = \frac{1}{s}\cancel{\frac{\partial M_z}{\partial \phi}}\hat{s} - \cancel{\frac{\partial M_z}{\partial s}}\hat{\phi}$$

$\Rightarrow \boxed{\vec{J}_b = 0}$ agrees!

c)



d)

$$\vec{K}_b = \vec{M} \times \hat{n} = \alpha s \, \hat{z} \times \hat{s} \Rightarrow \vec{K}_b = \alpha R \hat{\phi}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \cancel{\frac{1}{s} \frac{\partial M_z}{\partial \phi} \hat{s}} - \frac{\partial M_z}{\partial s} \hat{\phi}$$

$$= -\alpha \hat{\phi} \Rightarrow \vec{J}_b = -\alpha \hat{\phi}$$

e)

$$\vec{K}_b = \vec{M} \times \hat{n} = M_0 \cos\phi \, \hat{z} \times \hat{s} \Rightarrow \vec{K}_b = M_0 \cos\phi \, \hat{\phi}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{1}{s} \frac{\partial M_z}{\partial \phi} \hat{s} = -\frac{1}{s} \sin\phi \, \hat{s} \Rightarrow \vec{J}_b = -\frac{1}{s} \sin\phi \, \hat{s}$$