Phys 312 2025

Lecture 3

Tues: HW by 5pm

Fri: HW by 5pm

Thurs: Read 4.4.3, 6.1

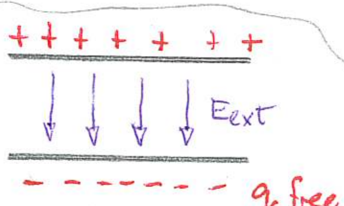
Thurs: Seminar

Fri: SPS

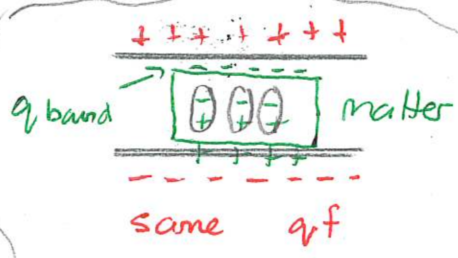
REU Info

Electrostatics in Matter

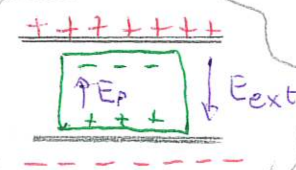
Materials that respond to external electric fields can produce their own electric fields. We aim to determine the total electric field in such situations using a model where the matter consists of electric dipoles. The scheme is



Free charges produce electric field $\vec{E}_{ext}$



Matter becomes polarized and this is described by a polarization vector $\vec{P}$. There will be bound charges on the matter



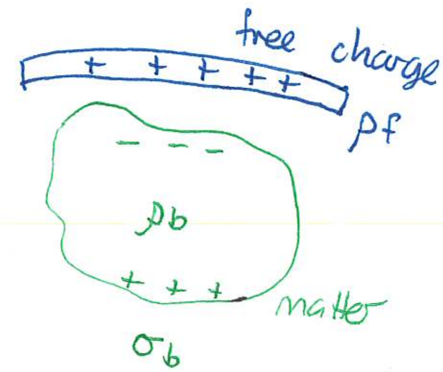
Bound charges produce electric field $\vec{E}_P$. The total electric field is

$$\vec{E} = \vec{E}_{ext} + \vec{E}_P$$

We aim for methods to compute the total electric field produced by the free and bound charges. This will be facilitated by an auxiliary quantity called the electric displacement

Electric displacement

Consider an arrangement of static free and band charges. Let ρ_f be the free charge density and ρ_b be the band charge density. Then the electric field is determined by the total charge density



$$\rho = \rho_f + \rho_b$$

The total electric field satisfies:

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla}\times\vec{E} = 0$$

Then

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho_f + \rho_b)$$

$$= \frac{1}{\epsilon_0}(\rho_f - \vec{\nabla}\cdot\vec{P}) \qquad \text{since } \rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$\Rightarrow \quad \vec{\nabla}(\vec{E} + \vec{P}/\epsilon_0) = \frac{1}{\epsilon_0}\rho_f$$

$$\Rightarrow \quad \vec{\nabla}(\epsilon_0\vec{E} + \vec{P}) = \rho_f$$

We define the electric displacement vector

$$\boxed{\vec{D} = \epsilon_0\vec{E} + \vec{P}} \qquad \text{units } C/m^2$$

Then

$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$$

Separately

$$\vec{\nabla}\times\vec{D} = \epsilon_0\cancel{\vec{\nabla}\times\vec{E}} + \vec{\nabla}\times\vec{P}$$

$$\Rightarrow \quad \boxed{\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}}$$

This gives a strategy

Knowing free charge density ρ_f, determine electric displacement $\vec{D}$ via

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

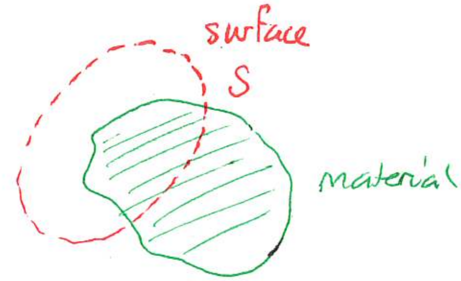
$\longrightarrow$ Knowing electric displacement and polarization $\vec{P}$ determine electric field

$$\vec{E} = \frac{1}{\epsilon_0}\left(\vec{D} - \vec{P}\right)$$

We now need strategies to determine the electric displacement. Any strategy that satisfies

$$\vec{\nabla}\cdot\vec{D} = \rho_f \quad \text{and} \quad \vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$$

will work. One strategy <u>always</u> works. This is a version of Gauss' Law. Here consider a closed region R bounded by surface S. Then



$$\int_R \vec{\nabla}\cdot\vec{D}\, d\tau' = \int \rho_f(\vec{r}\,')\, d\tau' = q_{\text{free enc}} \quad \leftarrow \text{total } \underline{\text{enclosed}}\ \underline{\underline{\text{free}}}\ \underline{\text{charge}}$$

$$\shortparallel$$

$$\int_S \vec{\nabla}\cdot d\vec{a}$$

These give

> For any closed surface S
>
> $$\oint_S \vec{D}\cdot d\vec{a} = q_{\text{free enc}}$$
>
> where $q_{\text{free enc}}$ is the total free charge enclosed by the surface

Additionally

> If $\vec{\nabla}\times\vec{P} = 0$ then any method used to determine electrostatic fields will work to determine $\vec{D}$ provided that we replace $\vec{E} \to \vec{D}/\epsilon_0$ and ρ by ρ_f

1 Electric field in a sphere with constant polarization

A sphere of radius R contains no free charge but has polarization

$$\mathbf{P}(\mathbf{r}') = P\,\hat{\mathbf{r}}$$

where $P > 0$ is a constant with units of $\mathrm{C/m^2}$.

a) Determine an expression for the electric displacement.

b) Determine an expression for the electric field inside and outside the sphere.

Answer: a) The polarization is spherically symmetric $\Rightarrow$ ρ_b spherically symmetric

$\Rightarrow \vec{E}$ spherically symmetric

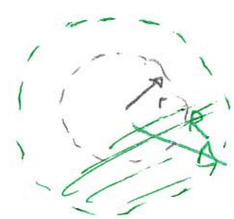
$\Rightarrow \vec{D}$ spherically symmetric

So

$$\oint_S \vec{D}\cdot d\vec{a} = q_{\text{free enc}}$$

for any enclosed surface. But $q_{\text{free enc}} = 0 \Rightarrow \oint_S \vec{D}\cdot d\vec{a} = 0$

The symmetry means $\vec{D} = D_r(r)\,\hat{r}$ and we choose as a Gaussian surface a sphere of radius r. Then on the sphere



$$\left.\begin{array}{l} r' = r \\ 0 \leq \theta' \leq \pi \\ 0 \leq \phi' \leq 2\pi \end{array}\right\} \quad d\vec{a}' = r'^2 \sin\theta'\, d\theta' d\phi'\,\hat{r}$$

$$= r^2 \sin\theta'\, d\theta' d\phi'\,\hat{r}$$

$$\Rightarrow \vec{D}\cdot d\vec{a} = r^2 \sin\theta' d\theta' d\phi'\; D_r(r)$$

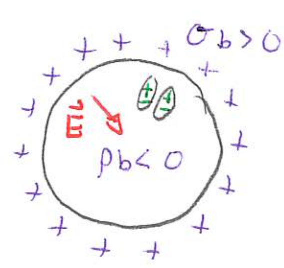
$$\Rightarrow \oint_S \vec{D}\cdot d\vec{a} = D_r(r)\, r^2 \int_0^{\pi} \sin\theta' d\theta' \int_0^{2\pi} d\phi' = 4\pi r^2 D_r(r) = 0$$

$$\Rightarrow D_r(r) = 0 \quad \Rightarrow \vec{D} = 0 \text{ everywhere.}$$

$\vec{E} = 0 \ldots$

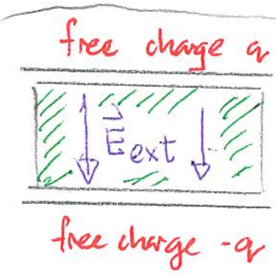
b)
$$\vec{E} = \frac{1}{\epsilon_0}\left[\vec{D} - \vec{P}\right]$$

$$= -\frac{1}{\epsilon_0}\vec{P} \quad \Rightarrow \quad \vec{E} = \begin{cases} -\frac{1}{\epsilon_0} P\hat{r} & \text{inside} \\ 0 & \text{outside} \end{cases}$$



Linear dielectrics

A crucial aspect of this process is to specify the polarization $\vec{P}$. If there is an external electric field we might expect



Free charge produces external field.

→ External field induces polarization in material

→ Polarization produces bound charges and these produce "internal" fields

→ Internal fields modify total electric field

adjusts polarization

One simple situation is where the polarization of the material is proportional to the total electric charge. Such a material is called a linear dielectric and here

$$\boxed{\vec{P} = \epsilon_0 \chi_e \vec{E}}$$

where $\vec{E}$ = field produced by all free and bound charges

χ_e = electric susceptibility of the material positive, no units

Then

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1+\chi_e) \vec{E}$$

and we define the permittivity of the material as:

$$\epsilon = \epsilon_0 (1+\chi_e) = \epsilon_0 \epsilon_r$$

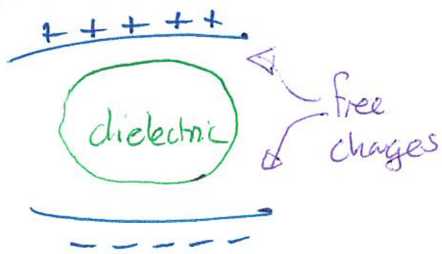
where the dielectric constant of the material is

$$\epsilon_r = (1+\chi_e)$$

Then

For a linear dielectric with permittivity ϵ, the electric field is determined from the displacement by $$\vec{E} = \frac{1}{\epsilon}\vec{D}$$

So, for a linear dielectric



Let ρ_f be the free charge density.

Electric displacement $\vec{D}$ satisfies
$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

Solve for $\vec{D}$

Get electric field
$$\vec{E} = \frac{1}{\epsilon}\vec{D}$$

Linear dielectric
$$\vec{D} = \epsilon\vec{E}$$
where $\epsilon = \epsilon_0\epsilon_r = \epsilon_0(1+\chi_e)$

If the linear dielectric is homogeneous, then χ_e and ϵ do not depend on location. Thus

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(\vec{D}/\epsilon\right) = \frac{1}{\epsilon}\vec{\nabla}\cdot\vec{D} = \frac{\rho_f}{\epsilon}$$

$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(\vec{D}/\epsilon\right) = \frac{1}{\epsilon}\left(\vec{\nabla}\times\vec{D}\right) = 0 \Rightarrow \vec{\nabla}\times\vec{D} = 0.$$

Thus for a linear homogeneous dielectric.

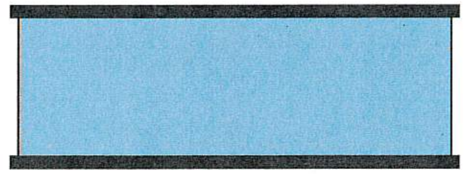
$$\vec{\nabla}\cdot\vec{E} = \rho_{free}/\epsilon$$

$$\vec{\nabla}\times\vec{E} = 0$$

The electric field can be determine from any technique of electrostatics with $\rho \to \rho_f$ and $\epsilon_0 \to \epsilon$.

2 Parallel plate capacitor filled with a dielectric

A parallel plate capacitor has plates of area A separated by distance d. The gap between the plates is filled with a dielectric material with permittivity ϵ. Assume that the gap between the plates is very small compared to their horizontal extent. The upper plate has uniform free surface charge density, $+\sigma_f$ and the lower plate $-\sigma_f$.



a) Determine an expression for the electric field between the capacitor plates.

b) Determine an expression for the electrostatic potential difference between the plates.

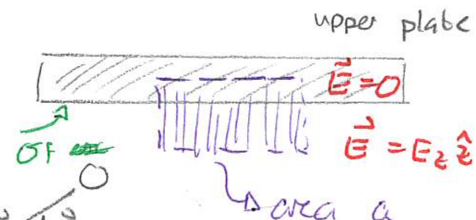
c) Determine an expression for the capacitance of the capacitor.

d) Suppose that initially the region between the plates is a vacuum but this is later filled by water (dielectric constant 80.4). By what factor will the capacitance change when the plates are filled by water?

e) If the free charge per unit area on the upper plate is $+\sigma_{\text{free}}$ and on the lower plate is $-\sigma_{\text{free}}$, determine expressions for the bound charge densities on the upper and lower surfaces of each dielectric.

Answer: a) The field satisfies $\vec{\nabla}\cdot\vec{E} = \rho_f/\epsilon \Rightarrow \oint_S \vec{E}\cdot d\vec{a} = q_{\text{free enc}}/\epsilon$

Use the illustrate Gaussian surface. By symmetric $\vec{E} = E_z(z)\hat{z}$



Then

$$\oint \vec{E}\cdot d\vec{a} = \underbrace{\cancel{\int \vec{E}\cdot d\vec{a}}}_{\substack{\text{upper surface}\\ \vec{E}=0}}^{0} + \underbrace{\int \vec{E}\cdot d\vec{a}}_{\text{lower surface}} + \underbrace{\cancel{\int \vec{E}\cdot d\vec{a}}}_{\substack{\text{sides}\\ \vec{E}\cdot d\vec{a}=0}}^{0}$$

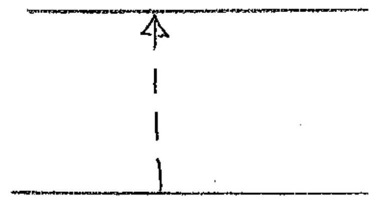
$$= -E_z(z)\, a$$

$$\Rightarrow \quad -E_z(z)a = \frac{q_{\text{free enc}}}{\epsilon} = \frac{\sigma_{\text{free enc}}\, a}{\epsilon\, a}$$

$$\Rightarrow \quad E_z(z) = -\frac{\sigma_{\text{free}}}{\epsilon} \qquad \Rightarrow \quad \vec{E} = \begin{cases} -\dfrac{\sigma_f}{\epsilon}\hat{z} & \text{between plates} \\ 0 & \text{outside} \end{cases}$$

b) In general $\Delta V = -\int \vec{E}\cdot d\vec{l}$

Use the illustrated path. Then



$$d\vec{l} = dz\,\hat{z}$$

and

$$V_{upper} - V_{lower} = -\int_{lower}^{upper} \vec{E}\cdot\hat{z}\,dz$$

$$= -\int -\frac{\sigma_f}{\epsilon}\,dz = \frac{\sigma_f}{\epsilon}d \Rightarrow \Delta V = \frac{\sigma_f}{\epsilon}d.$$

c) $Q = C\Delta V$ where $Q = \sigma_f A$ is the charge on the upper plate. So

$$\cancel{\sigma_f} A = C\frac{\cancel{\sigma_f}}{\epsilon}d \Rightarrow C = \epsilon\frac{A}{d}.$$

d) Without dielectric $\epsilon = \epsilon_0 \Rightarrow C_{without} = \epsilon_0 \frac{A}{d}$

With dielectric $\epsilon = \epsilon_r\epsilon_0 \Rightarrow C_{with} = \epsilon_0\epsilon_r A/d = \epsilon_r C_{without}$

80.4

Increases by a factor of 80.4

e) If we can determine $\vec{P}$ then

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$\sigma_b = \vec{P}\cdot\hat{n}$$

Now $\vec{D} = \epsilon\vec{E}$

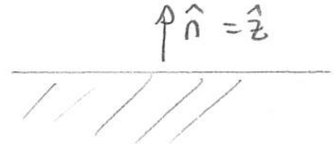
$$\epsilon_0\vec{E} + \vec{P} = \epsilon\vec{E} = \epsilon_0\epsilon_r\vec{E} \Rightarrow \vec{P} = \epsilon_0\epsilon_r\vec{E} - \epsilon_0\vec{E}$$

$$\Rightarrow \vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E}$$

Now $\vec{E}$ is uniform and thus $\vec{\nabla}\cdot\vec{E}=0$, so

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\epsilon_0(\epsilon_r-1)\cancel{\vec{\nabla}\cdot\vec{E}}\;0 \quad\Rightarrow\quad \rho_b = 0$$

On the upper surface, $\hat{n}=\hat{z}$ and thus



$$\sigma_b = \vec{P}\cdot\hat{n}$$

$$= \epsilon_0(\epsilon_r-1)\,\vec{E}\cdot\hat{n}$$

$$= \epsilon_0(\epsilon_r-1)\,\hat{z}\cdot\left(-\frac{\sigma_f}{\epsilon}\hat{z}\right) = -\frac{\epsilon_0(\epsilon_r-1)}{\epsilon}\sigma_f$$

But $\epsilon = \epsilon_0\epsilon_r$ gives

$$\sigma_b = -\left(\frac{\epsilon_r-1}{\epsilon_r}\right)\sigma_f = -\left(1-\frac{1}{\epsilon_r}\right)\sigma_f$$