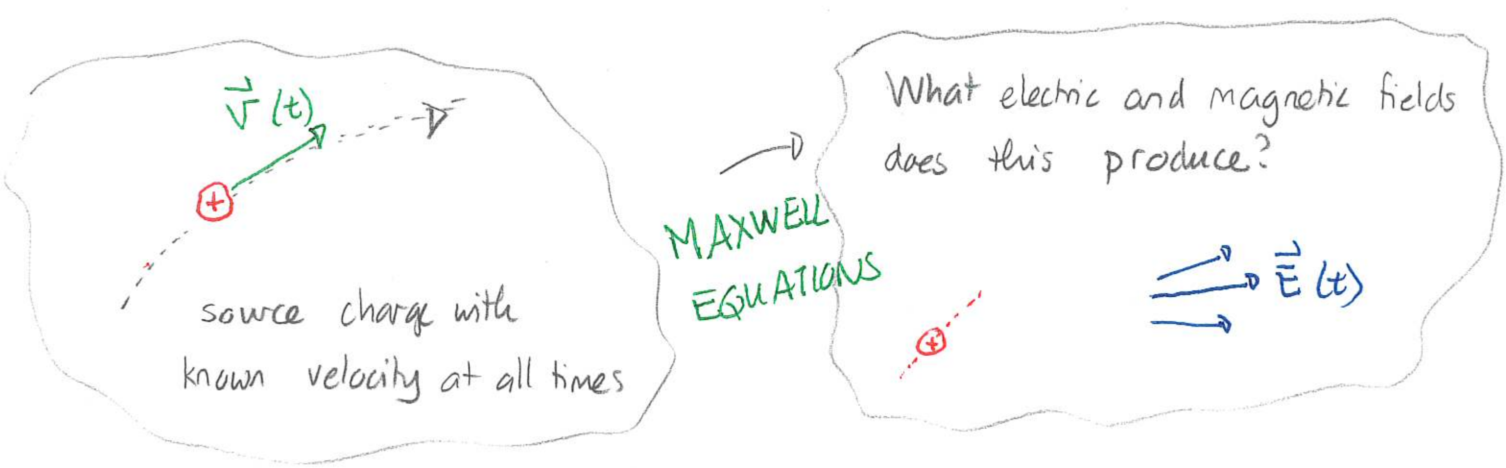
Lecture 1

* Handout - syllabus
* Course website
* Assignments - due Tuesdays / Fridays
  - first one Friday, Jan 24.
* Exams

Scope of the Course

Phys 312 is a continuation of Phys 311, which ended at Maxwell's equations. These plus the Lorentz force law are the key rules of classical electromagnetism. In Phys 311 you consider situations where the sources were stationary (did not vary with time). We will broaden this to consider more general situations, always starting with the basic Maxwell equations. The classic example is a moving point charge



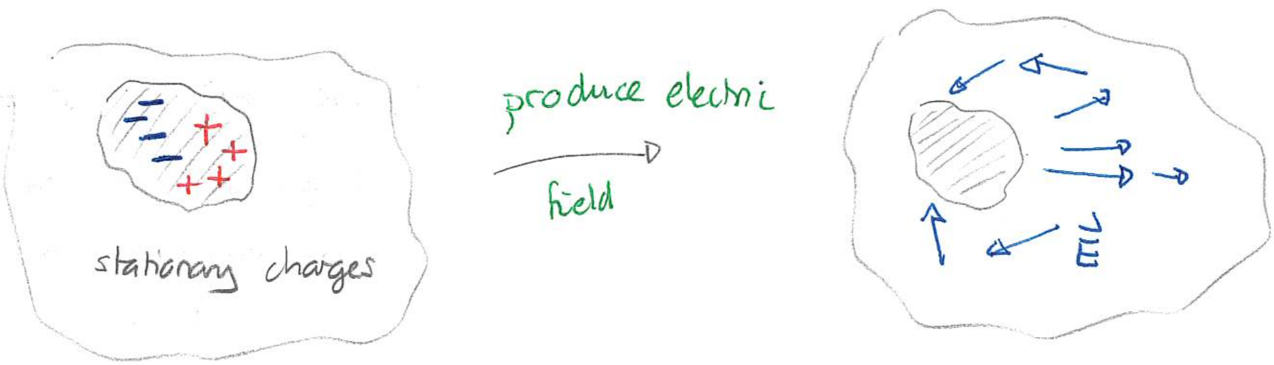
One of the predictions of this example is that the fields have a wavelike aspect. Such electromagnetic waves are also a direct consequence of Maxwell's equations.

Broadly the course will cover

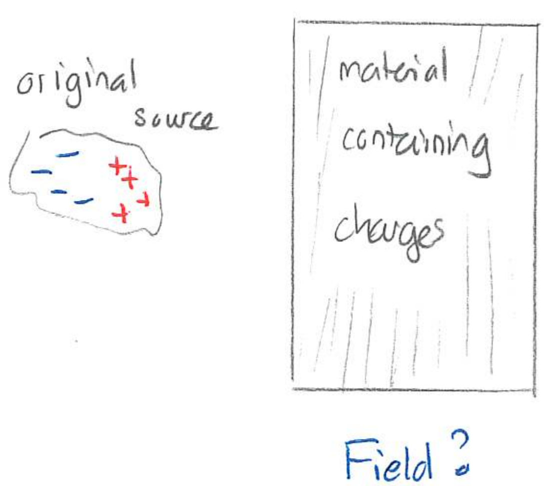
1) electric and magnetic fields in materials
2) electromagnetic waves
3) fields produced by moving point charges
4) fields in different reference frames and special relativity.

Electrostatics in matter

Electrostatics allows us to determine the electric field produced by a stationary charge distribution - this means a distribution which does not change as time passes. We might have



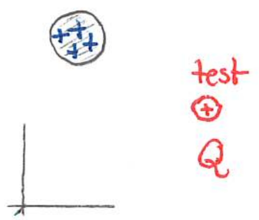
The question that arises is how this field might be modified if there were a material, itself containing charges that is present



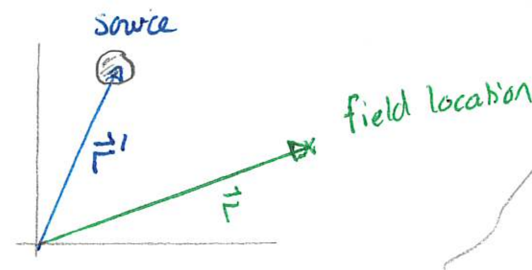
Review of electrostatics

The scheme for electrostatics is

Given a fixed charge distribution and a test charge



Sources produce an electric field $\vec{E} \equiv$ one vector at each location in space



Electric field satisfies (electrostatics)

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{E} = 0$$

where $\rho(\vec{r})$ is the charge density of the source charges

If a test particle with charge Q is placed at field location $\vec{r}$, then the force on this is

$$\vec{F} = Q E(\vec{r})$$

Electrostatics provides various ways to determine the electric field produced by stationary sources and these are all mathematical consequences of Maxwell's equations. The schemes use the following notation:

Let	$\vec{r} \equiv$ position vector that describes the location where field is calculated.
	$\vec{r}\,' \equiv$ position vector of contributing source charge.

The charge distribution is described via:

The charge density $\rho(\vec{r}\,')$, with units C/m^3, is a scalar depending on location $\vec{r}\,'$. The charge within an infinitesimal region of volume $d\tau'$ at $\vec{r}\,'$ is $dq' = \rho(\vec{r}\,')d\tau'$

There are various possible ways to find the electric field in electrostatics.

1) solving differential equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0} \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho(x,y,z)}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0$$

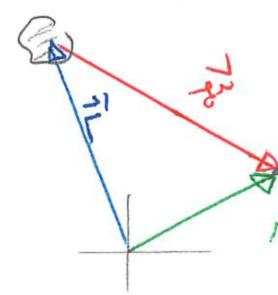
$$\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = 0$$

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = 0$$

Solve these!

2) using the integral form of Coulomb's Law:

$$\boxed{E(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\, d\tau'}$$



where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and

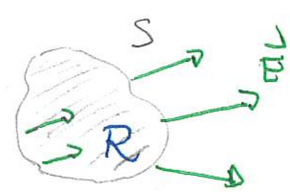
$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$$

One can verify that this satisfies $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ and $\vec{\nabla}\times\vec{E} = 0$ by direct differentiation.

3) using <u>Gauss' Law</u> which is a consequence of Maxwell's equations and the divergence theorem:

Let S be any closed surface and R the region that it encloses. Then

$$\oint_S \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$$



where $q_{enc} = \int_R \rho(\vec{r}\,')\,d\tau'$ is the total charge enclosed by S

1 Electric field produced by a charged sphere, 1

A solid sphere with radius R has charge density given in spherical coordinates by

$$\rho(\mathbf{r}') = \alpha r'$$

where α is a constant with units $\mathrm{C/m^4}$. There is no charge beyond the sphere.

a) Determine the total charge, Q, in the sphere. Use this to rewrite α in terms of Q.

b) Determine the electric field for $r \leqslant R$. Express the answer in terms of Q.

c) Determine the electric field for $R \leqslant r$. Express the answer in terms of Q.

d) Check that the resulting fields satisfy

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla \times \mathbf{E} = 0.$$

Answer: a) $Q = \int\limits_{\text{all space}} \rho(\vec{r}')\, d\tau'$

$$\underbrace{\begin{matrix} 0 \leq r' \leq R \\ 0 \leq \theta' \leq \pi \\ 0 \leq \phi' \leq 2\pi \end{matrix}}$$

$$d\tau' = r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'$$

$$\Rightarrow Q = \int_0^R dr' \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta'\, r'^2 \sin\theta'\, \alpha r'$$

$$= \alpha \underbrace{\int_0^R r'^3 dr'}_{R^4/4} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \underbrace{\int_0^{\pi} \sin\theta' d\theta'}_{2} = \alpha \pi R^4$$

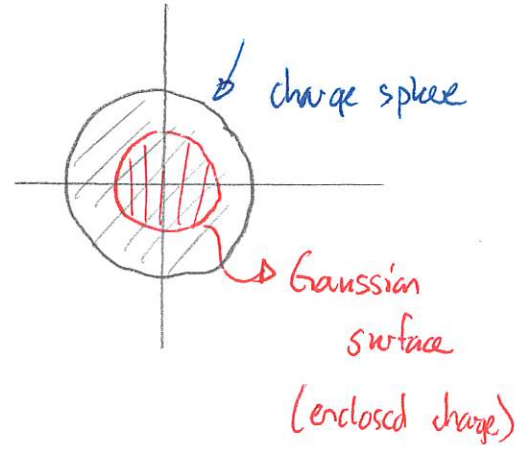
$$\Rightarrow \alpha = \frac{Q}{\pi R^4}$$

b) Symmetry implies that $\vec{E}$ only depends on r and only has an $\hat{r}$ component. So

$$\vec{E} = E_r(r)\, \hat{r}$$

Choose as a Gaussian surface a sphere of radius $r < R$. Then on the surface

$$\left.\begin{aligned} r &= r \\ 0 \le \theta &\le \pi \\ 0 \le \phi &\le 2\pi \end{aligned}\right\} \; d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$



Then

$$\vec{E} \cdot d\vec{a} = E_r(r)\,\hat{r} \cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$= r^2 E_r(r) \sin\theta \, d\theta \, d\phi .$$

and

$$\oint \vec{E} \cdot d\vec{a} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, r^2 E_r(r) = E_r(r)\, r^2 \, 4\pi$$

Thus

$$E_r(r)\, r^2 4\pi = q_{enc}/\epsilon_0 \quad = \quad E_r(r) = \frac{q_{enc}}{4\pi\epsilon_0 r^2}$$

Now to compute the enclosed charge,

$$q_{enc} = \int_{\text{insides}} \rho(\vec{r}')\, d\tau' \quad \text{and} \quad \left.\begin{aligned} 0 \le r' &\le r \\ 0 \le \theta' &\le \pi \\ 0 \le \phi' &\le 2\pi \end{aligned}\right\} \; d\tau' = r'^2 \sin\theta' dr' d\theta' d\phi'$$

gives

$$q_{enc} = \int_0^r dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \; r'^2 \sin\theta' \, \alpha r' = \alpha \underbrace{\int_0^r r'^3 dr'}_{r^4/4} \underbrace{\int_0^{\pi} \sin\theta' d\theta'}_{2} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi}$$

$$= \alpha \pi r^4$$

$$\Rightarrow \quad q_{enc} = \frac{Q}{\pi R^4} \pi r^4 = Q \frac{r^4}{R^4} \quad \Rightarrow \quad \boxed{E_r(r) = \frac{Q}{4\pi\epsilon_0} \frac{r^2}{R^4}}$$

c) This will be similar to b) except $Q_{enc} = Q$. Then

$$\boxed{E_r(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}}$$

Combining b), c) gives:

$$\boxed{\vec{E}(\vec{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{Qr^2}{R^4}\hat{r} & r \le R \\[2ex] \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\hat{r} & r \ge R \end{cases}}$$

d) We need $\vec{\nabla}\cdot\vec{E}$ in spherical co-ordinates:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, \cancel{E_\theta})^{0} + \frac{1}{r\sin\theta}\cancel{\frac{\partial E_\phi}{\partial\phi}}^{0}$$

Inside

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{Q}{4\pi\epsilon_0}\frac{r^4}{R^4}\right] = \frac{1}{r^2}\frac{Q}{4\pi\epsilon_0}\frac{1}{R^4}4r^3 = \frac{Q}{\pi\epsilon_0 R^4}r$$

$$= \rho(r)/\epsilon_0$$

Outside

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\epsilon_0}\right) = 0$$

Thus we see $\vec{\nabla}\cdot\vec{E} = \rho(\vec{r})/\epsilon_0$

Separately

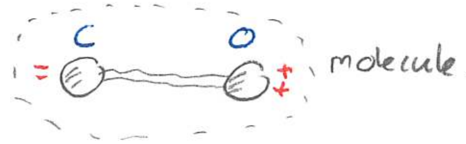
$$\vec{\nabla}\times\vec{E} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,\cancel{E_\phi})^{0} - \cancel{\frac{\partial E_\theta}{\partial\phi}}^{0}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial E_r}{\partial\phi}}^{0} - \frac{\partial}{\partial r}(r\,\cancel{E_\phi})^{0}\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,\cancel{E_\theta}) - \cancel{\frac{\partial E_r}{\partial\theta}}^{0}\right]\hat{\phi} = 0$$

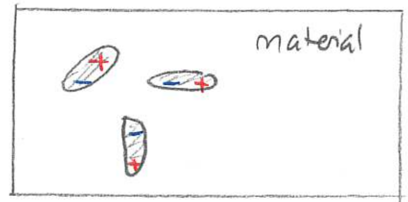
Thus $\vec{\nabla}\times\vec{E} = 0$.

Electric dipoles

An electric dipole is a distribution of charge where one end is predominantly positive and the other is predominantly negative while the total charge is zero. Dipoles occur in molecules in nature. For example, carbon monoxide has an asymmetrical charge distribution.



We will create models of matter in which the material is a collection of electric dipoles and we will use this model to describe how the material responds to and modifies the electric field produced by external source charges.

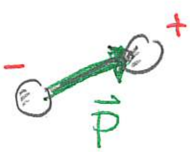


In general we can use electrostatics to describe how a dipole:

1) produces its own electric field
2) responds to external electric fields.

Electrostatics provides:

The dipole can be described via a dipole moment



calculated via

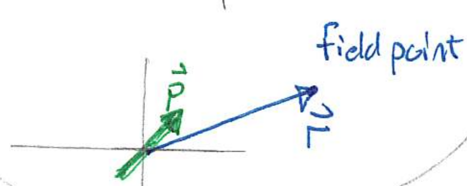
$$\vec{p} = \int_{\text{all space}} \rho(\vec{r}\,')\,\vec{r}\,'\,d\tau'$$

point charge distribution:

$$\vec{p} = \sum q_i \vec{r}_i\,'$$

The electrostatic potential produced by a dipole, situated at the origin, at field point $\vec{r}$ is

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}$$



The electric field produced by the dipole is

$$\vec{E} = -\vec{\nabla} V_{dip}$$

2 Electric dipole field

A point electric dipole has dipole moment

$$\mathbf{p} = p\hat{\mathbf{z}}$$

where p has units of $\mathrm{C}\cdot\mathrm{m}$. Using spherical coordinates, determine the electric field produced by this dipole.

Answer:

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

Now $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\Rightarrow \hat{r}\cdot\hat{z} = \cos\theta \qquad \Rightarrow \qquad \boxed{V_{dip} = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}}$

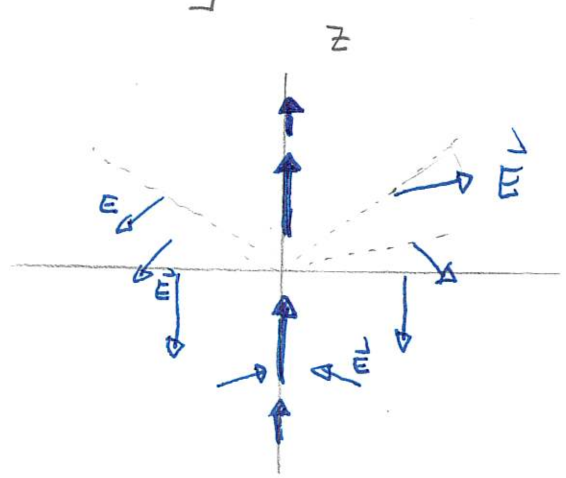
Now

$$\vec{E} = -\vec{\nabla} V_{dip}$$

$$= -\left[\frac{\partial V_{dip}}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V_{dip}}{\partial\theta}\hat{\theta} + \cancel{\frac{1}{r\sin\theta}\frac{\partial V_{dip}}{\partial\phi}\hat{\phi}}\right]$$

$$= -\frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^3}(-2)\hat{r} - \frac{1}{r}\frac{p}{4\pi\epsilon_0}\frac{-\sin\theta}{r^2}\hat{\theta}$$

$$= \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$



DEMO: Show dipole field PhET

We can show that in general, for a point dipole at the origin

$$\boxed{\vec{E}_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\ 3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\ \right]}$$

Derivation of the dipole potential

The potential due to any charge distribution at point $\vec{r}$ is

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{\gamma}\,d\tau'$$

where $\vec{\gamma} = \vec{r}-\vec{r}'$. Then

$$\gamma = \sqrt{(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')} = \left(r^2 + r'^2 - 2\vec{r}'\cdot\vec{r}\right)^{1/2}$$

$$= r\left(1 + \frac{r'^2}{r^2} - \frac{2\vec{r}'\cdot\vec{r}}{r^2}\right)^{1/2}$$

implies

$$V = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \frac{\rho(\vec{r}')}{\left(1 + r'^2/r^2 - \frac{2\vec{r}'\cdot\vec{r}}{r^2}\right)^{1/2}}\,d\tau'$$

Then the Taylor series gives:

$$\left(1 + \frac{r'^2}{r^2} - 2\frac{\vec{r}'\cdot\vec{r}}{r^2}\right)^{-1/2} = \left(1+x\right)^{-1/2} \qquad \text{where} \quad x = \frac{r'^2}{r^2} - 2\frac{\vec{r}'\cdot\vec{r}}{r^2}$$

$$\approx 1 - \frac{1}{2}x + \frac{3}{4}x^2 + \dots$$

For $r \gg r'$ $x \gg 1$ and the series can be terminated at the "x" term.

So

$$\left(1 + \frac{r'^2}{r^2} - \frac{2\vec{r}'\cdot\vec{r}}{r^2}\right)^{-1/2} = 1 - \frac{1}{2}\left[\frac{r'^2}{r^2} - \frac{2\vec{r}'\cdot\vec{r}}{r^2}\right] + \text{smaller}.$$

So

$$V \approx \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left\{\int \rho(\vec{r}')d\tau' + \int \frac{\vec{r}'\cdot\vec{r}\,\rho(\vec{r}')d\tau'}{r^2} - \frac{1}{2}\int \frac{r'^2\rho(\vec{r}')}{r^2}\,d\tau' + \dots\right\}$$

Now

$$\int \rho(\vec{r}')\, d\tau' = Q$$

which is the total charge in the distribution.

Then

$$\frac{\vec{r}' \cdot \vec{r}}{r^2} = \frac{\vec{r}'}{r} \cdot \hat{r} = \left(\frac{\vec{r}'}{r}\right) \cdot \hat{r}$$

and this is larger than $\left(\frac{r'}{r}\right)^2$. So

$$V \approx \frac{1}{4\pi\epsilon_0} \frac{1}{r} \left\{ Q + \int \hat{r} \cdot \frac{\vec{r}'}{r} \rho(\vec{r}')\, d\tau' + \cdots \right.$$

$$= \underbrace{\frac{Q}{4\pi\epsilon_0 r}}_{\substack{\text{monopole} \\ \text{contribution/term} \\ V_{mon}}} + \underbrace{\frac{1}{4\pi\epsilon_0 r^2} \hat{r} \cdot \int \vec{r}' \rho(\vec{r}')\, d\tau'}_{\substack{\text{dipole contribution/term} \\ V_{dip}}} + \cdots$$

Then define the dipole moment as:

$$\vec{p} = \int \rho(\vec{r}') \vec{r}'\, d\tau'$$

and

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$