Electromagnetic Theory II: Homework 12

Due: 11 March 2025

1 General plane wave solution to the three dimensional wave equation

a) Show that

$$f(\mathbf{r}, t) = g(\mathbf{k} \cdot \mathbf{r} \pm \omega t),$$

where $g(u)$ is any twice differentiable function of a single variable, is a solution to the three dimensional wave equation.

b) Suppose that

$$g(u) = \frac{A}{u^2 + b^2}$$

where A and b are constants. If $\mathbf{k} = k\left(\hat{\mathbf{x}} + 2\hat{\mathbf{y}}\right)/\sqrt{5}$ and $b = 2$, sketch (in the xy plane)

- the location of the maximum displacement at $t = 0$ and
- the locations along which the displacement is half of the maximum value at $t = 0$.

In which direction does the wave propagate?

2 Three dimensional waves

Consider the following candidates for three dimensional functions:

$$\begin{aligned}
f_1(\mathbf{r}, t) &:= A(x + 2y - vt)e^{-(x+2y-vt)^2/a^2} \\
f_2(\mathbf{r}, t) &:= A(x - vt)e^{-(x+2y-vt)^2/a^2} \\
f_3(\mathbf{r}, t) &:= A(x^2 + y^2 - vt)e^{-(x^2+y^2-vt)^2/a^2} \\
f_4(\mathbf{r}, t) &:= A\cos\left(x + y - vt\right) \\
f_5(\mathbf{r}, t) &:= A\cos\left(x^2 + y^2 - vt\right)
\end{aligned}$$

where $a > 0$.

a) Explain which of these might represent plane waves and which do not represent plane waves.

b) For those which are do represent plane waves, identify the wavenumber vector $\mathbf{k}$.

3 Spherical waves

Consider the three dimensional wave equation

$$\nabla^2 f = \frac{1}{v^2}\,\frac{\partial^2 f}{\partial t^2}.$$

In spherical coordinates, this gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$

Assume a spherically symmetrical solution of the form

$$f(\mathbf{r},t)=\frac{h(r)}{r}e^{-i\omega t}$$

By substituting, determine a differential equation for $h(r)$ and solve this for $h(r)$ (use the complex representation when solving this).

4 Electromagnetic waves

Write the real and complex representations of the electric and magnetic fields for the following sinusoidal plane electromagnetic waves in a vacuum.

a) Traveling along the $+y$ direction with electric field amplitude E_0, frequency ω, phase $\delta = 0$ and polarization along the $+x$ direction.

b) Traveling along the $+y$ direction with electric field amplitude E_0, frequency ω, phase $\delta = 0$ and polarization along the $+z$ direction.

c) Traveling along in the xy plane at an angle of $60°$ counterclockwise from the $+x$ axis and with electric field amplitude E_0, frequency ω, phase $\delta = 0$ and polarization along the $+z$ direction.