Electromagnetic Theory II: Homework 8

Due: 18 February 2025

1 Electrostatic and magnetostatic energy for a rotating charge distribution.

A sphere with radius R has a uniform surface charge density σ and rotates about an axis through its center with angular velocity ω. The magnetic vector potential for this distribution can be calculated exactly (see Example 5.11).

a) Determine the magnetic field inside and outside the sphere by using the result of Example 5.11.

b) Determine the electric field inside and outside the sphere by using the surface charge density.

c) *(Optional: maximum of 4 bonus points.)* With moving charges, it is not immediately clear that the fields can be calculated using electrostatics and magnetostatics. However, if that method was used, one can determine whether the resulting fields satisfy Maxwell's equations (for the given charge and current distributions). Check that the fields obtained in the previous parts do satisfy Maxwell's equations.

d) Determine the total energy stored in the electric field.

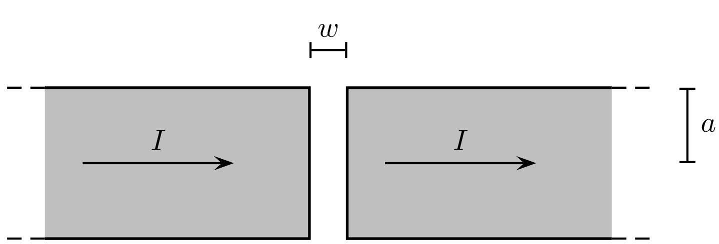
e) Determine the total energy stored in the magnetic field.

f) Determine the ratio between the two energies and express this in terms of the speed of light $c = 1/\sqrt{\mu_0 \epsilon_0}$ and the speed of a point on the equator of the sphere.

g) Which form of energy dominates for rotational speeds much less than the speed of light?

2 Electromagnetic fields in a gap between two conductors

An infinitely long perfectly conducting cylinder, whose cross section has radius a, is cut so that there is a gap, which is small compared to its radius. This is as illustrated.



A constant current, I, which was turned on at $t = 0$ and which is uniformly distributed across the conductor flows as illustrated.

a) Determine expressions for the volume charge and current densities within the conductors and the gap. Determine an expression for the surface charge density, $\sigma(t)$, on the left face of the gap in terms of I and the parameters for the dimensions of the conductors.

b) Determine expressions for both the electric and magnetic fields within the conductors (note that the gap is small enough that you could assume that each conductor is infinitely long.) Verify that both are independent of time.

c) Using the expression for the charge density on the face of the gap, and ignoring any possible contribution from time-varying magnetic fields, determine an expression for the electric field within the gap. Verify that this is time-dependent.

d) Starting with the equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

find an expression for the magnetic field within the gap. Verify that this is time-independent.

e) *(Optional: maximum of 4 bonus points.)* You should now have expressions for the electric fields and magnetic fields in the conductor and in the gap. However, these were solved in a piecewise fashion; particularly a contribution from the electric field due to a time-varying magnetic field was not considered. Verify by direct substitution that your expressions for the fields in the gap satisfy Maxwell's equations (using the charge and current densities for the gap). Repeat this for the fields in the conductors.

f) *(Optional: maximum of 2 bonus points.)* It is still possible that the fields in each part could be correct but that they could not match appropriately. Verify that the solutions satisfy the boundary conditions on the left face of the gap. If, so you have found the correct fields.

g) Determine an expression for the energy density and the Pointing vector within the gap. Show that they satisfy

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{S}$$

h) Use the Poynting vector to determine the rate at which energy is transported by the field through the gap.