Electromagnetic Theory II: Class Exam I

25 February 2025

Name: Solution **Total:** /50

Instructions

- There are 6 questions on 7 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

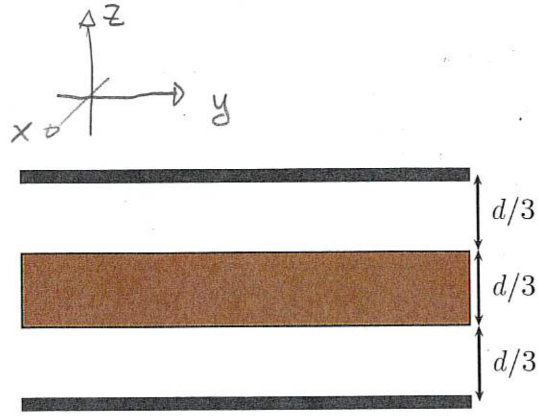
$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1



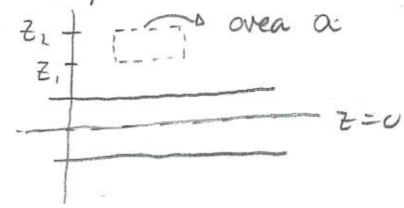
A parallel plate capacitor consists of two conducting plates separated by distance d. A linear dielectric, with permittivity ϵ, occupies the middle third of the region between its plates. A cross-sectional view of the arrangement is as illustrated. The area of the plates is A and the gap between the plates is sufficiently small for them to be considered infinite in extent.

7 a) Suppose that the capacitor plates are equally and oppositely charged. The free surface charge density on the upper plates is $+\sigma > 0$ and on the lower plate it is $-\sigma$. Determine $\mathbf{D}$ for all regions and use this to determine the electric field at all locations between the plates.

In general $\oint_{\text{surface}} \vec{D} \cdot d\vec{a} = q_{\text{free enc}}$. Then $\vec{D} = D_x \hat{x} + D_y \hat{y} + D_z \hat{z}$

Using the co-ordinate system above, symmetry under $180°$ rotations about $\hat{z}$ gives $D_x = D_y = 0$. Thus $\vec{D} = D_z(z)\hat{z}$.

We then use a pillbox with sides either parallel or perpendicular to $\hat{z}$. First use a pillbox outside the plates. Then $q_{\text{free enc}} = 0$



$$\Rightarrow \oint \vec{D} \cdot d\vec{a} = 0$$

$$\Rightarrow \int_{\text{top}} \vec{D} \cdot d\vec{a} + \int_{\text{bottom}} \vec{D} \cdot d\vec{a} + \int_{\text{sides}} \vec{D} \cdot d\vec{a} = 0$$

On the sides $d\vec{a}$ is perpendicular to $\vec{D}$ $\Rightarrow \int_{\text{sides}} \vec{D} \cdot d\vec{a} = 0$. On the top $d\vec{a} = da\,\hat{z}$ $\Rightarrow \int_{\text{top}} \vec{D} \cdot d\vec{a} = D_z(z_2)\,a$

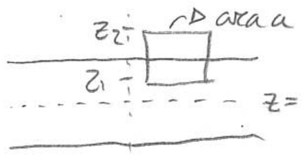
On the bottom $d\vec{a} = -da\,\hat{z}$ $\Rightarrow \int_{\text{bottom}} \vec{D} \cdot d\vec{a} = -D_z(z_1)\,a$.

Thus $D_z(z_2)a - D_z(z_1)a = 0 \Rightarrow D(z_2) = D(z_1)$ above

As $z_2 \to \infty$ $D \to 0$. Thus outside $\vec{D} = 0$

Question 1 continued ...

To get $\vec{D}$ inside use this pillbox



$$\oint \vec{D}\cdot d\vec{a} = \sigma_f a$$

$$\Rightarrow \underbrace{D_z(z_2)}_{0 \text{ outside}}\cancel{a} - D_z(z_1)\cancel{a} = \sigma_f \cancel{a}$$

$$\Rightarrow D_z(z_1) = -\sigma_f$$

$\Rightarrow$ Inside $\boxed{\vec{D} = -\sigma_f \hat{z}}$

In free space $\vec{E} = \vec{D}/\epsilon_0$

In material $\vec{E} = \vec{D}/\epsilon$.

Thus

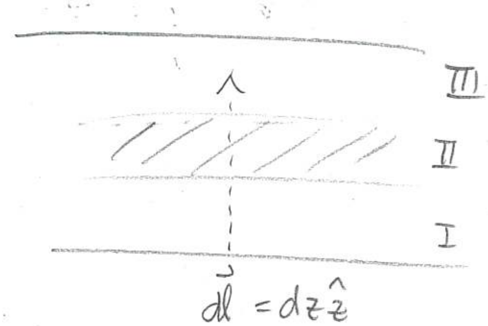
$$\vec{E} = \begin{cases} 0 & \text{outside} \\ -\dfrac{\sigma_f}{\epsilon_0}\hat{z} & \text{outside material between plates} \\ -\dfrac{\sigma_f}{\epsilon}\hat{z} & \text{inside material} \end{cases}$$

b) Determine the capacitance of this arrangement. 4 5

$$Q = C\Delta V \Rightarrow \boxed{\sigma_f A = C\Delta V}$$

We need $\Delta V = -\int_{\text{path}} \vec{E}\cdot d\vec{l}$

Use the illustrated path.



$$\Delta V = -\int_{I} \vec{E}\cdot d\vec{l} - \int_{II} \vec{E}\cdot d\vec{l} - \int_{III} \vec{E}\cdot d\vec{l}$$

$$= -\int_{I} -\frac{\sigma}{\epsilon_0}\,dz - \int_{II} -\sigma/\epsilon\,dz - \int_{III} -\frac{\sigma}{\epsilon_0}\,dz.$$

$$= \frac{\sigma}{\epsilon_0}\frac{d}{3} + \frac{\sigma}{\epsilon}\frac{d}{3} + \frac{\sigma}{\epsilon_0}\,d/3 = \frac{\sigma d}{3}\left(\frac{2}{\epsilon_0} + \frac{1}{\epsilon}\right)$$

So

$$\cancel{\sigma_f} A = C\,\frac{\cancel{\sigma_f} d}{3}\left(\frac{2\epsilon + \epsilon_0}{\epsilon_0\epsilon}\right) \Rightarrow \boxed{C = \frac{3A\epsilon_0\epsilon}{d(2\epsilon + \epsilon_0)}}$$

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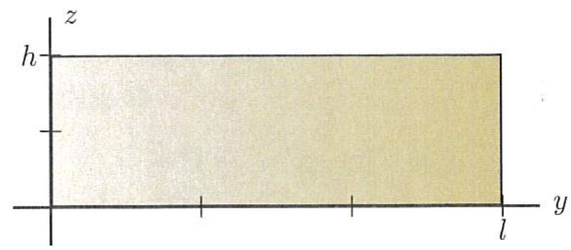
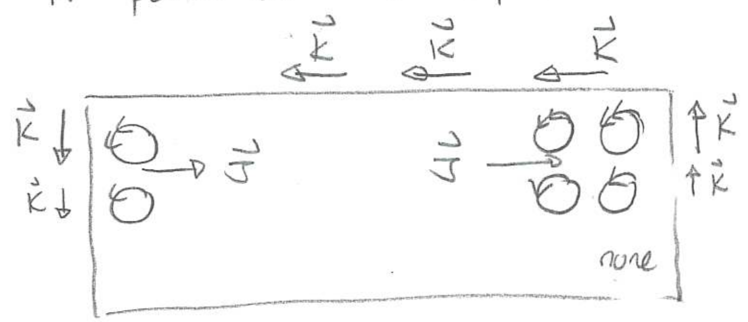
Question 2

A slab of material occupies the region $0 \leqslant x \leqslant w$, $0 \leqslant y \leqslant l$, and $0 \leqslant z \leqslant h$. The magnetization within the material is $\mathbf{M} = kz\hat{\mathbf{x}}$ where $k > 0$.



a) Suppose that the magnetization were produced by miniscule magnetic dipoles. Sketch some of these dipoles and use your sketch to predict the directions of the surface and volume bound currents.

$\vec{M}$ points out $\Rightarrow$ dipole currents are ↺



b) Determine expressions for the bound surface (on each surface) and volume current densities.

$\vec{K}_b = \vec{M} \times \hat{n}$ where $\hat{n}$ is outward normal

surface	$\hat{n}$	$\vec{K}_b$
$z=h$	$\hat{z}$	$-kh\hat{y}$
$z=0$	$-\hat{z}$	0 since $z=0$
$y=l$	$\hat{y}$	$kz\,\hat{z}$
$y=0$	$-\hat{y}$	$-kz\,\hat{z}$
$x=w$	$\hat{x}$	0
$x=0$	$-\hat{x}$	0

$$\vec{J}_b = \vec{\nabla}\times\vec{M} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kz & 0 & 0 \end{vmatrix} = 0\hat{x} + \frac{\partial}{\partial y}(kz)\,\hat{y} + 0\vec{z}$$

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$\Rightarrow \vec{J}_b = k\hat{y}$

Question 3

A point dipole, with dipole moment $\mathbf{p}$ is placed in an electric field $\mathbf{E} = E_0 x\,\hat{\mathbf{z}}$ where $E_0 > 0$ is a constant. Which of the following (choose one) is true about the force $\mathbf{F}$ exerted by the field on the dipole?

i) If $\mathbf{p}$ is along $\hat{\mathbf{x}}$ then $\mathbf{F} = 0$.
ii) If $\mathbf{p}$ is along $\hat{\mathbf{x}}$ then $\mathbf{F} = F\hat{\mathbf{z}} \neq 0$.
iii) If $\mathbf{p}$ is along $\hat{\mathbf{x}}$ then $\mathbf{F} = F\hat{\mathbf{x}} \neq 0$.
iv) If $\mathbf{p}$ is along $\hat{\mathbf{z}}$ then $\mathbf{F} = F\hat{\mathbf{z}} \neq 0$.
v) If $\mathbf{p}$ is along $\hat{\mathbf{z}}$ then $\mathbf{F} = F\hat{\mathbf{x}} \neq 0$.

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$$
$$= p_x \frac{\partial}{\partial x}\vec{E} + \cancel{p_y \frac{\partial}{\partial y}\vec{E}} + \cancel{p_z \frac{\partial}{\partial z}\vec{E}}$$
$$= p_x E_0 \hat{z}$$

so $\vec{F} = F\hat{z}$ if $p_x \neq 0$

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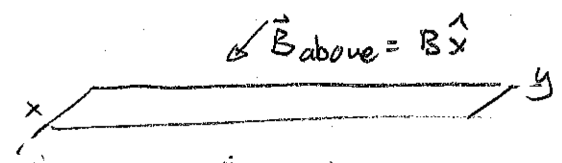
Question 4

A rectangular piece of linear magnetic material has a surface in the xy-plane. There is free space above the surface. The magnetic field immediately above the surface is $\mathbf{B}_{\text{above}} = B\hat{\mathbf{x}}$ where $B > 0$. There is a surface current in the surface along $+\hat{\mathbf{y}}$. Which of the following (choose one) is true about the magnetic field immediately below the plane, $\mathbf{B}_{\text{below}}$?

i) $B_{\text{below}\ z} = 0$ and $B_{\text{below}\ y} = 0$.
ii) $B_{\text{below}\ z} = 0$ and $B_{\text{below}\ y} \neq 0$.
~~iii)~~ $B_{\text{below}\ z} \neq 0$ and $B_{\text{below}\ y} = 0$.
~~iv)~~ $B_{\text{below}\ z} \neq 0$ and $B_{\text{below}\ y} \neq 0$.



Briefly explain your choice.

The boundary conditions are

$$B_{above}^{\perp} = B_{below}^{\perp} \Rightarrow \underbrace{B_{above\,z}}_{=0} = B_{below\,z} \Rightarrow B_{below\,z} = 0$$

and

$$\vec{B}_{above}^{\parallel} - \vec{B}_{below}^{\parallel} = \mu_0 \vec{K}\times\hat{n} = \mu_0 K\hat{y}\times\hat{z} = \mu_0 K\hat{x}$$
$$B\hat{x} - \vec{B}_{below}^{\parallel} = \mu_0 K\hat{x}$$

so $\vec{B}_{below}^{\parallel}$ is only in the x direction

$\Rightarrow B_{below\,y} = 0$

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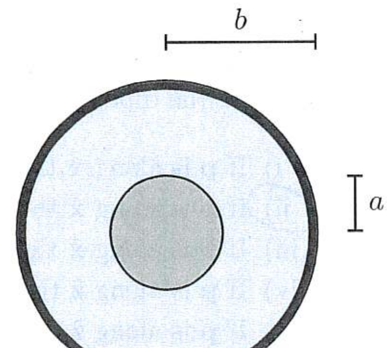
Question 5

A coaxial arrangement consists of two cylinders separated by a region filled with a linear magnetic material with magnetic susceptibility χ_m. The inner cylinder has radius a and the outer cylinder has radius b. The inner cylinder carries a uniformly distributed current I flowing out of the page. The outer cylinder carries a uniformly distributed current I into the page. Determine the magnetic field $\mathbf{B}$, in terms of I, χ_m, μ_0, and radial distance, at all points beyond the inner cylinder.



$$\underset{\text{closed loop}}{\oint} \vec{H} \cdot d\vec{l} = I_{\text{free enc.}}$$

Here $\vec{H} = H_s \hat{s} + H_\phi \hat{\phi} + H_z \hat{z}$. By Biot-Savart Law $H_z = 0$

By inversion about a transverse axis $H_s = 0$. So $\vec{H} = H_\phi(s)\hat{\phi}$.

Use a loop with radius $s > a$ centered on the axis.

$$\left.\begin{array}{l} s' = s \\ 0 \le \phi' \le 2\pi \\ z' = \text{const} \end{array}\right\} \quad d\vec{l} = s' d\phi' \hat{\phi} = s\, d\phi' \hat{\phi}$$

$$\Rightarrow \oint \vec{H} \cdot d\vec{l} = \int_0^{2\pi} d\phi' s H_\phi(s) = 2\pi s H_\phi(s) = I_{\text{free enc}}.$$

If $a < s < b$ then $I_{\text{free enc}} = I \Rightarrow \vec{H} = \frac{I}{2\pi s}\hat{\phi}$ inside

If $b < s$ then $I_{\text{free enc}} = 0 \Rightarrow \vec{H} = 0$ outside

Now $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ and $\vec{M} = \chi_m \vec{H}$

$$\Rightarrow \vec{H}(1 + \chi_m)\mu_0 = \vec{B}$$

$$\Rightarrow \boxed{\vec{B} = \begin{cases} \dfrac{I\mu_0(1+\chi_m)}{2\pi s}\hat{\phi} & a < s < b \\ 0 & b < s \end{cases}}$$

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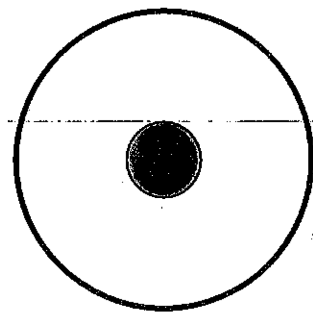
Question 6

Two cylindrical conductors are concentric and arranged as illustrated (viewed along their axis). The radius of the inner conductor is a and of the outer conductor is b. The fields in the gap are



$$\mathbf{E} = \frac{\alpha}{s}\hat{\mathbf{s}} \quad \text{and} \quad \mathbf{B} = \frac{\beta}{s}\hat{\phi}$$

where $\alpha, \beta > 0$. Determine the direction of flow of electromagnetic energy in the region between the cylinders and determine the total energy that flows per second through a closed cylindrical surface, whose axis is along that of the conductor's axis and which has radius $a \leqslant r \leqslant b$ and length L. *Note: The cylinder has three surfaces.*

Energy flow is along $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{\alpha\beta}{s^2}\hat{s}\times\hat{\phi}$

$\Rightarrow \vec{S} = \frac{\alpha\beta}{\mu_0 s^2}\hat{z}$ $\rightarrow$ flow is along $\hat{z}$

The flow out of the cylinder is $\oint \vec{S}\cdot d\vec{a}$. The cylinder has three sides. Then on the curved side $d\vec{a} = ds\,\hat{s} \Rightarrow \vec{S}\cdot d\vec{a} = 0$

$$\int_{\text{curved}} \vec{S}\cdot d\vec{a} = 0$$

On the top $d\vec{a} = s'ds'd\phi'\,\hat{z}$

$$\Rightarrow \int_{\text{top}} \vec{S}\cdot d\vec{a} = \int_0^{2\pi} d\phi' \int_a^r ds\, s\,\frac{\alpha\beta}{\mu_0 s^2} = \frac{2\pi\alpha\beta}{\mu_0}\ln\left(\frac{r}{a}\right)$$

On the bottom $d\vec{a} = -s'ds'd\phi'\hat{z}$

$$\Rightarrow \int_{\text{bottom}} \vec{S}\cdot d\vec{a} = -\frac{2\pi\alpha\beta}{\mu_0}\ln(r/a)$$

$$\Rightarrow \oint \vec{S}\cdot d\vec{a} = 0$$

So the total energy flowing through is zero

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