Fri: HW by 5pm

Tues: Read

Divergence theorem

Surface integrals are defined by:

1) specifying a vector field $\vec{v}$
2) specifying a surface → can use a two-parameter parametrization

→ construct area vectors $d\vec{a}$ perpendicular to surface.

Then we can integrate over the entire surface to obtain

$$\int_{\text{surface}} \vec{v}\cdot d\vec{a}.$$

Strictly if we parameterize the surface with two parameters, say s,t.

$$\int \vec{v}\cdot d\vec{a} = \iint \Big\{ V_x(x(s,t),y(s,t),z(s,t)) \left[\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial z}{\partial s}\frac{\partial y}{\partial t}\right] + V_y\left[\frac{\partial z}{\partial s}\frac{\partial x}{\partial t} - \frac{\partial x}{\partial s}\frac{\partial z}{\partial t}\right] + V_z\left[\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right]\Big\}\, ds\, dt.$$

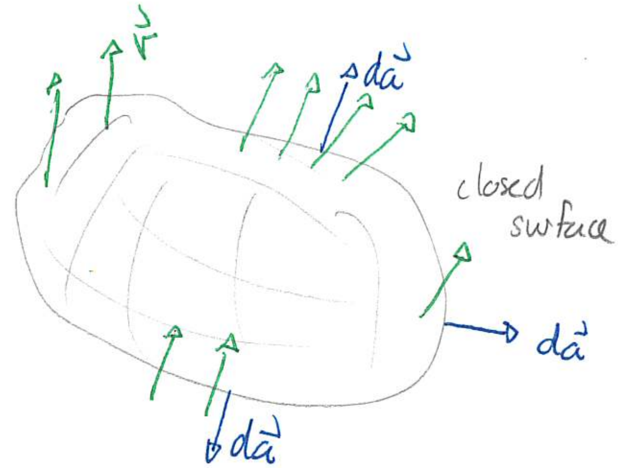
Alternatively, if for example x,y are appropriate parameters and the surface is described by $z = h(x,y)$ then use

$$d\vec{a} = -\left[\frac{\partial h}{\partial x}\hat{x} + \frac{\partial h}{\partial y}\hat{y} - \hat{z}\right]dx\,dy$$

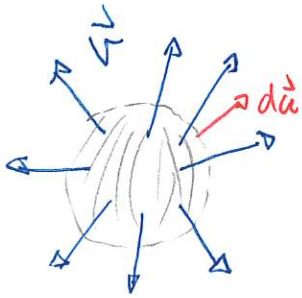
We often integrate over a closed surface. In that case the notation for the surface integral is

$$\oint \vec{v} \cdot d\vec{a}$$

and we arrange for $d\vec{a}$ to point outwards.

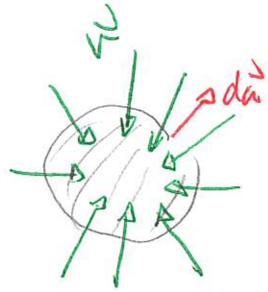


We will encounter vector fields that diverge or converge on a source or sink



$\vec{v} \cdot d\vec{a} > 0$ everywhere

$\Rightarrow \oint \vec{v} \cdot d\vec{a} > 0$



$\vec{v} \cdot d\vec{a} < 0$ everywhere

$\Rightarrow \oint \vec{v} \cdot d\vec{a} < 0$

Notice $\vec{\nabla} \cdot \vec{v} > 0$ $\qquad$ $\vec{\nabla} \cdot \vec{v} < 0$

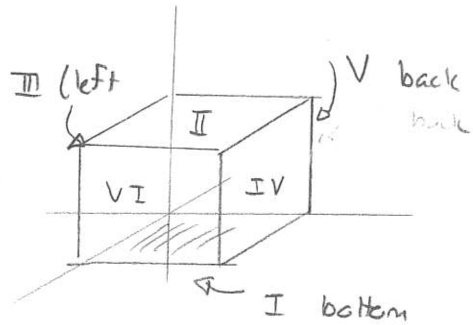
In general such closed surface integrals are related to divergences via the divergence theorem.

> For any closed surface S which bounds a region $\mathcal{V}$
>
> $$\underset{\substack{S \\ \text{surface}}}{\oint} \vec{v} \cdot d\vec{a} = \underset{\substack{\mathcal{V} \\ \text{region}}}{\int} \vec{\nabla} \cdot \vec{v} \, d\tau$$

1 Divergence theorem

Let $\mathbf{v} = x^2y^2\hat{\mathbf{y}}$ and verify the divergence theorem over the cube that bounds $0 \leqslant x, y, z, \leqslant a$.

Answer: Surface integral → there are six sides



Side	x	y	z	$d\vec{a}$	$\vec{v}\cdot d\vec{a}$
I	$0 \to a$	$0 \to a$	0	$-dxdy\hat{z}$	0
II	$0 \to a$	$0 \to a$	a	$dxdy\hat{z}$	0
III	$0 \to a$	0	$0 \to a$	$-dxdz\hat{y}$	$0\,dxdz = 0$
IV	$0 \to a$	a	$0 \to a$	$dxdz\hat{y}$	x^2a^2dxdz
V	0	$0 \to a$	$0 \to a$	$-dydz\hat{x}$	0
VI	a	$0 \to a$	$0 \to a$	$dydz\hat{x}$	0

So $$\int_{IV} \vec{v}\cdot d\vec{a} = \int_0^a dx \int_0^a dz\, x^2a^2 = a^2 \int_0^a x^2 dx \int_0^a dz = \frac{a^2a^3}{3}a = \frac{a^6}{3}$$

$$\Rightarrow \oint \vec{v}\cdot d\vec{a} = \frac{a^6}{3}$$

They match

Volume integral requires $\vec{\nabla}\cdot\vec{v} = \frac{\partial v_y}{\partial y} = 2yx^2$

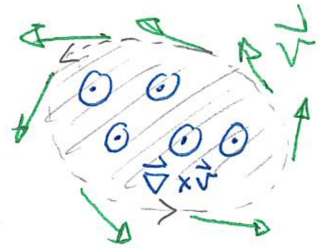
$$\left.\begin{array}{l} 0 \leq x \leq a \\ 0 \leq y \leq a \\ 0 \leq z \leq a \end{array}\right\} \Rightarrow d\tau = dxdydz$$

$$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{v}\, da = \int_0^a dx \int_0^a dy \int_0^a dz\, 2yx^2 = 2\int_0^a x^2 dx \int_0^a y\,dy \int_0^a dz = 2\,\frac{a^3}{3}\,\frac{a^2}{2}\cdot a = \frac{a^6}{3}$$

Stoke's theorem

Now consider a line integral of a circling field. Suppose that the path circles in the same sense as the field and completes the loop.

We aim to evaluate



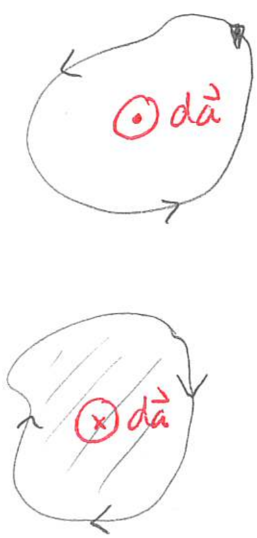
$$\oint \vec{v}\cdot d\vec{l}$$

Such fields are not the gradient of a function and we cannot use the fundamental theorem for gradients.

The origin of this is $\vec{\nabla}\times\vec{v} \neq 0$. We can see that for the surface enclosed by the loop

$$\int \vec{\nabla}\times\vec{v}\cdot d\vec{a} \neq 0$$

Stoke's theorem provides an exact relationship. First we provide an area vector convention related to the loop direction. Then



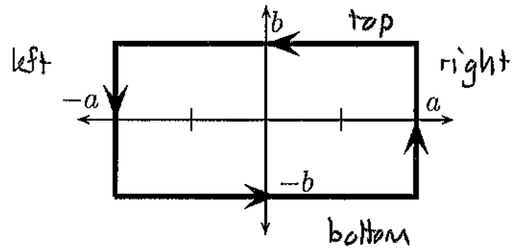
right hand rule.

> Let S be any surface with a boundary loop. Then
>
> $$\int_{\text{surface}} \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \oint_{\text{loop}} \vec{v}\cdot d\vec{l}$$

2 Stokes' theorem, 1

Let

$$\mathbf{v} = -\frac{y}{2}\,\hat{\mathrm{x}} + \frac{x}{2}\,\hat{y}.$$



a) Determine $\oint \mathbf{v}\cdot d\mathbf{l}$ around the loop.

b) Determine $\int \nabla\times\mathbf{v}\cdot d\mathbf{a}$ for the flat surface enclosed by the loop. Verify Stokes' theorem.

Answer: a)

$$\oint \vec{v}\cdot d\vec{l} = \int_{\text{right}} \vec{v}\cdot d\vec{l} + \int_{\text{top}} \vec{v}\cdot d\vec{l} + \int_{\text{left}} \vec{v}\cdot d\vec{l} + \int_{\text{bottom.}} \vec{v}\cdot d\vec{l}$$

Right edge

$$\left.\begin{array}{l} x = a \\ -b \le y \le b \end{array}\right\} \; d\vec{l} = dy\,\hat{y} \quad \Rightarrow \quad \vec{v}\cdot d\vec{l} = \left(-\frac{y}{2}\hat{x} + \frac{a}{2}\hat{y}\right)\cdot \hat{y}\,dy$$

$$= \frac{a}{2}\,dy$$

$$\Rightarrow \int_{\text{right}} \vec{v}\cdot d\vec{l} = \frac{a}{2}\int_{-b}^{b} dy = ab$$

Top edge

$$\int_{\leftarrow} \vec{v}\cdot d\vec{l} = -\int_{\rightarrow} \vec{v}\cdot d\vec{l} \qquad \text{So evaluate} \quad \int_{\rightarrow} \vec{v}\cdot d\vec{l}$$

Here

$$\left.\begin{array}{l} y = b \\ -a \le x \le a \end{array}\right\} \Rightarrow d\vec{l} = dx\,\hat{x} \quad \Rightarrow \quad \vec{v}\cdot d\vec{l} = -\frac{b}{2}\,dx$$

$$\Rightarrow \int_{\rightarrow} \vec{v}\cdot d\vec{l} = -\frac{b}{2}\int_{-a}^{a} dx = -ab \quad \Rightarrow \quad \int_{\leftarrow} \vec{v}\cdot d\vec{l} = ab$$

Left edge

$$\int_{\downarrow} \vec{v}\cdot d\vec{l} = -\int_{\uparrow} \vec{v}\cdot d\vec{l}$$

then for $\uparrow\int$

$$\left.\begin{array}{c} x=-a \\ -b \le y \le b \end{array}\right\} \quad d\vec{l} = dy\,\hat{y} \qquad \vec{v}\cdot d\vec{l} = -\frac{a}{2}\,dy$$

$$\int \vec{v}\cdot d\vec{l} = \int_{-b}^{b} -\frac{a}{2}\,dy = -ab \Rightarrow \int_{\downarrow} \vec{v}\cdot d\vec{l} = a\,b$$

Bottom

$$\left.\begin{array}{c} y=-b \\ -a \le x \le a \end{array}\right\} \Rightarrow d\vec{l} = +\frac{b}{2}\,dx$$

$$\int \vec{v}\cdot d\vec{l} = \frac{b}{2}\int_{-a}^{a} dx = ab$$

Thus $\oint \vec{v}\cdot d\vec{l} = 4ab.$

b) $$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y/2 & x/2 & 0 \end{vmatrix} = \hat{z}$$

For the integral

$$\left.\begin{array}{c} -a \le x \le a \\ -b \le y \le b \end{array}\right\} \Rightarrow d\vec{a} = dx\,dy\,\hat{z}$$

$$\vec{\nabla}\times\vec{v}\cdot d\vec{a} = dx\,dy$$

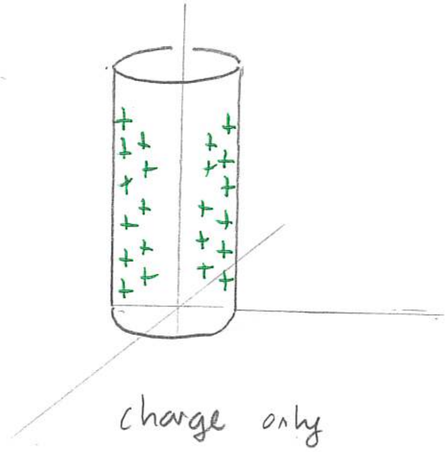
$$\int \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \int_{-a}^{a} dx \int_{-b}^{b} dy = 2a\,2b \Rightarrow \int \vec{\nabla}\times\vec{v}\cdot d\vec{a} = 4ab$$

MATCHES!

Cylindrical co-ordinates

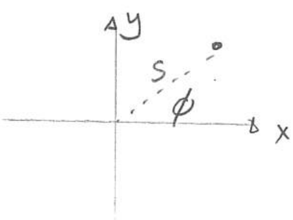
There will be physical situations which are symmetrical under rotations about one axis. For example, consider charge distributed on a cylinder with axle along the z-axis.

It will be less convenient to use Cartesian co-ordinates than co-ordinates adapted to a cylindrical situation. These are defined via:



charge only depends on the radial distance.

$$\boxed{\begin{aligned} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= z \end{aligned}}$$



with $0 \leq s \leq \infty$

$0 \leq \phi \leq 2\pi$

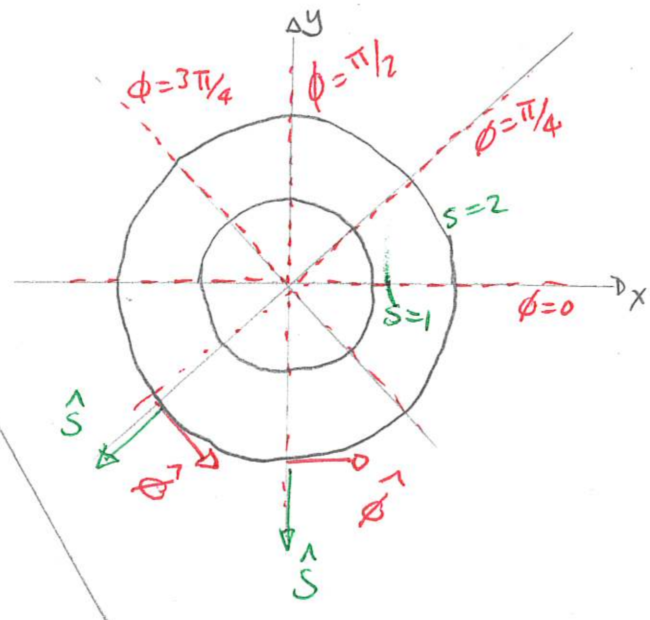
These are equivalent to

$$\begin{aligned} s &= \sqrt{x^2+y^2} \\ \phi &= \arctan(y/x) \\ z &= z \end{aligned}$$

These will result in:

1) special co-ordinate systems and unit vectors for cylindrical co-ordinates
2) cylindrical line elements
3) cylindrical area elements
4) cylindrical volume elements
5) cylindrical versions of gradients, divergences and curls, ...

We can envision a co-ordinate grid in the x-y plane. The unit vectors are:



$\hat{s}$ = tangent to lines where ϕ is constant in direction of increasing s

$\hat{\phi}$ = tangent to lines where s is constant in direction of increasing ϕ

Note that these vectors vary from one location to another, unlike Cartesian basis vectors. One can relate these to Cartesian basis vectors by geometry and trigonometry.

$$\boxed{\begin{aligned} \hat{s} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \\ \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \\ \hat{z} &= \hat{z} \end{aligned}} \iff \boxed{\begin{aligned} \hat{x} &= \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi} \\ \hat{y} &= \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi} \\ \hat{z} &= \hat{z} \end{aligned}}$$

They satisfy

$$\begin{array}{lll} \hat{s}\cdot\hat{s} = 1 & \hat{s}\cdot\hat{\phi} = 0 & \hat{s}\times\hat{\phi} = \hat{z} \\ \hat{\phi}\cdot\hat{\phi} = 1 & \hat{s}\cdot\hat{z} = 0 & \hat{\phi}\times\hat{z} = \hat{s} \\ \hat{z}\cdot\hat{z} = 1 & \hat{\phi}\cdot\hat{z} = 0 & \hat{z}\times\hat{s} = \hat{\phi} \end{array}$$

Any vector can be expressed as

$$\vec{V} = V_s(s,\phi,z)\,\hat{s} + V_\phi(s,\phi,z)\,\hat{\phi} + V_z(s,\phi,z)\,\hat{z}$$

Now consider calculus on such vectors. The line element transforms as

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$= \left(\frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial z} dz\right)\hat{x}$$

$$+ \left(\frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial \phi} d\phi + \frac{\partial y}{\partial z} dz\right)\hat{y}$$

$$+ \ldots$$

$$= \left[\cos\phi\, ds + s\sin\phi\, d\phi\right]\left[\cos\phi\,\hat{s} - \sin\phi\,\hat{\phi}\right]$$

$$\left[\sin\phi\, ds + s\cos\phi\, d\phi\right]\left[\sin\phi\,\hat{s} + \cos\phi\,\hat{\phi}\right] + dz\,\hat{z}$$

$$= ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

Thus

line element $\vec{dl} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

surface elements

$z =$ constant	$d\vec{a} = \pm s\,ds\,d\phi\,\hat{z}$
$s =$ constant	$d\vec{a} = s\,d\phi\,dz\,\hat{s}$
$\phi =$ constant	$d\vec{a} = ds\,dz\,\hat{\phi}$

volume element

$$d\tau = s\,ds\,d\phi\,dz$$

Gradient, divergence and curl

It may seem that

$$\cancel{\vec{\nabla} g = \frac{\partial g}{\partial s}\hat{s} + \frac{\partial g}{\partial \phi}\hat{\phi} + \frac{\partial g}{\partial z}\hat{z}}$$

$$\cancel{\vec{\nabla}\cdot\vec{v} = \frac{\partial V_s}{\partial s} + \frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}}$$

These are not true since the position dependent basis vectors must be included in the differentiation. We can chain calculus together to get.

$$\vec{\nabla} g = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z}$$

$$= \left(\frac{\partial g}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial x}\right)\hat{x} + \left(\frac{\partial g}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial y}\right)\hat{y} + \frac{\partial g}{\partial z}\hat{z}$$

$$= \frac{\partial g}{\partial s}\left[\frac{\partial s}{\partial x}\hat{x} + \frac{\partial s}{\partial y}\hat{y}\right] + \frac{\partial g}{\partial \phi}\left(\frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y}\right) + \frac{\partial g}{\partial z}\hat{z}$$

$$= \frac{\partial g}{\partial s}\left[\frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\hat{y}\right] + \cdots$$

$$= \frac{\partial g}{\partial s}\left[\frac{s}{s}\cos\phi\,\hat{x} + \frac{s}{s}\sin\phi\,\hat{y}\right] + \cdots$$

$$= \frac{\partial g}{\partial s}\hat{s} + \cdots$$

$$\Rightarrow \quad \vec{\nabla} g = \frac{\partial g}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial g}{\partial \phi}\hat{\phi} + \frac{\partial g}{\partial z}\hat{z}$$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(sV_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

etc...

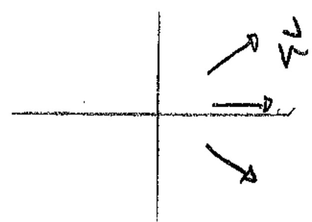
3 Divergence theorem, cylindrical coordinates

Consider $\mathbf{v} = \hat{\mathbf{s}}$.

a) Do you expect $\nabla \cdot \mathbf{v}$ to be zero or not?

b) Show that the divergence theorem is valid for a cylinder with axis along the z-axis, radius R and height h.

Answer a) In the x-y plane it appears that $\vec{v}$ diverges.



$$\vec{\nabla}\cdot\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s V_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

So $V_s = 1$, $V_\phi = 0$, $V_z = 0$ $\Rightarrow$ $\vec{\nabla}\cdot\vec{v} = \frac{1}{s}$

b)

$$\oint \vec{v}\cdot d\vec{a} = \underbrace{\int \vec{v}\cdot d\vec{a}}_{\text{curved surface}} + \underbrace{\int \vec{v}\cdot d\vec{a}}_{\text{top}} + \underbrace{\int \vec{v}\cdot d\vec{a}}_{\text{bottom}}$$

here $d\vec{a} = \pm s\,ds\,d\phi\,\hat{z}$

$\Rightarrow \vec{v}\cdot d\vec{a} = 0$

here

$$\left.\begin{matrix} s = R \\ 0 \le \phi \le 2\pi \\ 0 \le z \le h \end{matrix}\right\} \Rightarrow d\vec{a} = s\,d\phi\,dz\,\hat{s} = R\,d\phi\,dz\,\hat{s}$$

$$\Rightarrow \int \vec{v}\cdot d\vec{a} = \int_0^{2\pi} d\phi \int_0^h dz\, R = 2\pi R h$$

Then

$$\int \vec{\nabla}\cdot\vec{v}\, d\tau = \int \frac{1}{s}\underbrace{d\tau}_{d\tau = s\,ds\,d\phi\,dz} = \int_0^h dz \int_0^{2\pi} d\phi \int_0^R s\,ds\,\frac{1}{s} = 2\pi R h$$

MATCH!