Phys 311 2025

Lecture 5

Tues: HW ~~4~~ by 5pm HW2 corrections

~~Thurs~~: Fri: HW5 by 5pm

Thurs: Read

Integration theorems for line integrals

Calculus provides convenient methods for computing integrals of functions. The fundamental theorem of calculus states:

For any suitably well-behaved function $f(x)$,

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

Thus evaluating

$$\int g(x)\, dx$$

reduces to finding the anti-derivative of $g(x)$, i.e. $f(x)$ so that $g = \frac{df}{dx}$. If we can find such $f(x)$ then we can use the fundamental theorem of calculus to yield

$$\int_a^b g(x)\, dx = f(b) - f(a)$$

We consider this in the context of line integrals. We will see:

* it is not always possible to evaluate a line integral via an antiderivative

* there is a simple condition that describes when it is possible to evaluate a line integral via an antiderivative.

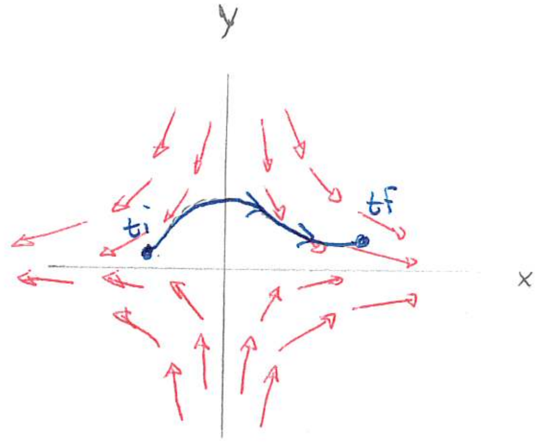
Recall that a line integral involves:

1) a vector field

$$\vec{v} = v_x(x,y,z)\,\hat{\imath} + v_y(x,y,z)\,\hat{\jmath} + v_z(x,y,z)\,\hat{k}$$

2) a path parameterized by a variable t

$$x(t),\ y(t),\ z(t)$$



e.g. $\vec{v} = \frac{x}{2}\hat{x} - \frac{y}{2}\hat{y}$

Then the line integral of $\vec{v}$ along the path for $t_i \le t \le t_f$ is

$$\boxed{\int \vec{v}\cdot d\vec{l} = \int_{t_i}^{t_f} \left[v_x \frac{dx}{dt} + v_y \frac{dy}{dt} + v_z \frac{dz}{dt} \right] dt}$$

Note that $v_x \equiv v_x(x(t), y(t), z(t))$, etc,... are functions of t. The question is now

"Given $\vec{v}$ does there exist some function $g(x,y,z)$ so that

$$\int_{t_i}^{t_f} \vec{v}\cdot d\vec{l} = g(x_f, y_f, z_f) - g(x_i, y_i, z_i)$$

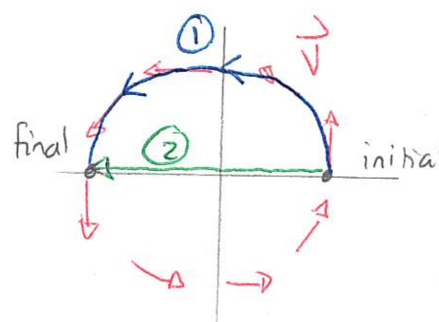
where $x_i = x(t_i)$ etc... "
$y_i = y(t_i)$

The function does not refer to path used in the integral. So such a function will only exist if the line integral is path-independent. We know that this is not always true. For example, with

$$\vec{v} = -y\hat{x} + x\hat{y}$$

and the two illustrated paths

$$\int_{①} \vec{v}\cdot d\vec{l} > 0 \quad \text{and} \quad \int_{②} \vec{v}\cdot d\vec{l} = 0$$



$V = -y\hat{x} + x\hat{y}$

So this line integral does depend on the path and it is not possible to find the special function g.

Fundamental theorem for gradients

One situation where we can easily evaluate line integrals is if there exists a scalar function $g(x,y,z)$ such that

$$\vec{v} = \vec{\nabla} g$$

Then

$$\vec{v} = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z}$$

and

$$\int_{t_i}^{t_f} \vec{v}\cdot d\vec{l} = \int_{t_i}^{t_f}\left[\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{\partial t} + \frac{\partial g}{\partial z}\frac{dz}{dt}\right]dt$$

$$= \int_{t_i}^{t_f} \frac{d}{dt}\, g(x(t), y(t), z(t))\, dt$$

Fundamental theorem of calculus

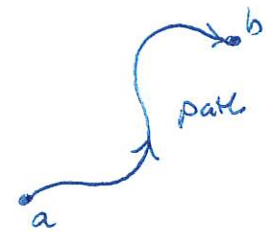
$$= g(x(t_f), y(t_f), z(t_f)) - g(x(t_i), y(t_i), z(t_i))$$

$$= g(\text{end of path}) - g(\text{start of path})$$

This is true regardless of the path. This partially proves: the <u>fundamental theorem for gradients</u>

Given a vector field $\vec{v}$ there exists a function g such that $\vec{v} = \vec{\nabla} g$ if and only if

$$\int_{\text{path}} \vec{v}\cdot d\vec{l} = \int_{\text{path}} \vec{\nabla} g \cdot d\vec{l} = g(b) - g(a)$$



where $g(b) = g(x,y,z)\big|_{\text{evaluate at end } b}$

$g(a) = g(x,y,z)\big|_{\text{evaluate at end } a}$.

We have proved the "only if" direction. Consider the reverse, that there exists a function g such that

$$\int \vec{v} \cdot d\vec{l} = g(b) - g(a)$$

for any path. Consider the path

$$x(t) = s \qquad 0 \leq s \leq t$$
$$y(t) = 0$$
$$z(t) = 0$$

Then

$$\int \vec{v} \cdot d\vec{l} = g(x(t), 0, 0) - g(0,0,0)$$

$$\Rightarrow \int_0^t \left[V_x \underbrace{\frac{dx}{ds}}_{1} + V_y \cancel{\frac{dy}{ds}} + V_z \cancel{\frac{dz}{ds}} \right] ds = g(x(t), 0, 0) - g(0,0,0)$$

$$\Rightarrow \int_0^t V_x \, ds = g(x(t), 0, 0)$$

Differentiate w.r.t. t

$$\Rightarrow \frac{\partial}{\partial t} \int_0^t V_x \, ds = \frac{\partial g}{\partial x} \qquad \Rightarrow \qquad V_x(x(t), y(t), z(t)) = \frac{\partial g}{\partial x}$$

continuing gives $\vec{v} = \vec{\nabla} g$. ∎

Some consequences of the fundamental theorem for gradients are:

① If $\vec{v} = \vec{\nabla} g$ then $\int \vec{v} \cdot d\vec{l}$ only depends on the initial and final points and not the path

② If $\vec{v} = \vec{\nabla} g$ then around any closed loop $\oint_{loop} \vec{v} \cdot d\vec{l} = 0$.

If around any closed loop $\oint \vec{v} \cdot d\vec{l} = 0$ then $\vec{v} = \vec{\nabla} g$

Note that if $\vec{v} = \vec{\nabla} g$ then $\vec{\nabla} \times \vec{v} = 0$. Thus if

$$\int \vec{v} \cdot d\vec{l} = g(b) - g(a) \text{ then } \vec{\nabla} \times \vec{v} = 0$$

Then converse is also true. So

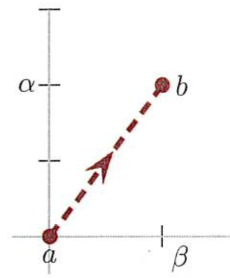
There exists g such that

$$\int \vec{v} \cdot d\vec{l} = g(b) - g(a)$$

regardless of path $\Leftrightarrow$ $\vec{\nabla} \times \vec{v} = 0$

1 Fundamental theorem for gradients

Let $g(x, y, z) = \tau x^2 y$ where τ is constant.



a) Determine $\mathbf{v} := \boldsymbol{\nabla} g$.

b) Determine $\int \mathbf{v} \cdot \mathrm{d}\mathbf{l}$ along the illustrated path.

c) Determine $g(b) - g(a)$ and check that the fundamental theorem for gradients is valid.

Answer: a) $\vec{v} = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} = 2\tau xy\,\hat{x} + \tau x^2\,\hat{y}$

b) The path can be described by parameter x

$$\left.\begin{array}{l} 0 \leq x \leq \beta \\ y = \frac{\alpha}{\beta} x \end{array}\right\} \Rightarrow d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$$

$$= dx\left[\hat{x} + \frac{dy}{dx}\hat{y}\right]$$

$$\Rightarrow d\vec{l} = \left(\hat{x} + \frac{\alpha}{\beta}\hat{y}\right)dx$$

So $\vec{v} \cdot d\vec{l} = \left(v_x + v_y\, \alpha/\beta\right) dx$

$$= \left(2\tau x y + \tau x^2 \alpha/\beta\right)dx$$

$\hookrightarrow \alpha/\beta\, x$

$$= \left(2\tau\, \alpha/\beta\, x^2 + \tau x^2 \alpha/\beta\right)dx = 3\tau \alpha/\beta\, x^2 dx.$$

$$\int \vec{v}\cdot d\vec{l} = \int_0^\beta 3\tau \alpha/\beta\, x^2 dx = \frac{3\tau\alpha}{\beta}\frac{1}{3}x^3\Big|_0^\beta = \tau\alpha\beta^2$$

c) $g(b) = \tau\beta^2\alpha$

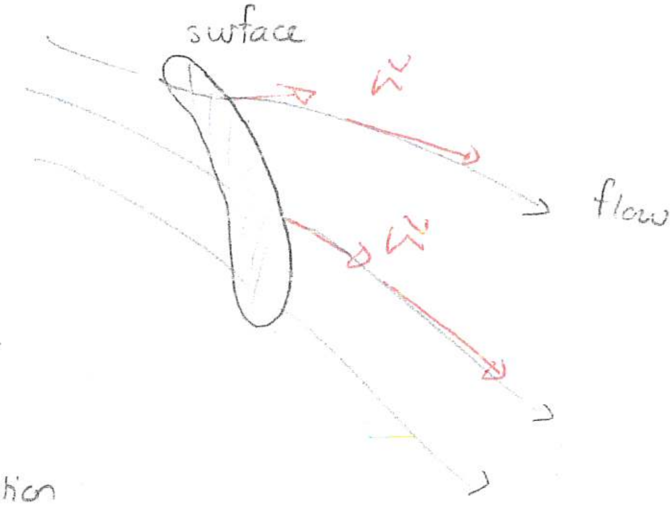
$g(a) = 0$

We see $\int \vec{v}\cdot d\vec{l} = g(b) - g(a)$.

Surface integrals

We will also need to develop the notion of a flow or flux associated with a vector field. We illustrate this via fluid flow.

Example: Consider a fluid that flows in three dimensions. We would like to quantify the rate at which fluid volume passes through any imaginary surface. Specifically we want the volume of fluid that passes every second. This is called the (volume) flux of the fluid. This depends on



1) the surface – size, shape, orientation

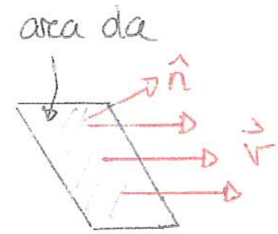
2) the way in which the fluid flows.

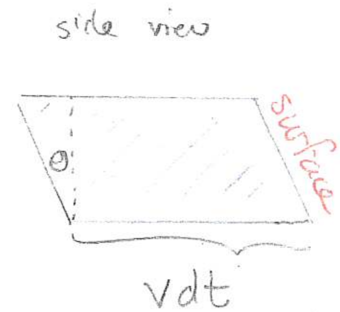
The fluid flow is described via a velocity vector at each location

$\vec{v}$ = velocity vector field

The calculation can be simplified by considering a small flat, planar fragment of surface across which the velocity vector is uniform

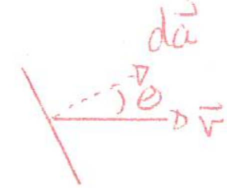
The shaded volume will all pass in time dt.





This volume is

$$dV = v\,dt\,da\cos\theta$$



and the volume rate of flow is

$$\frac{dV}{dt} = v\cos\theta\, da$$

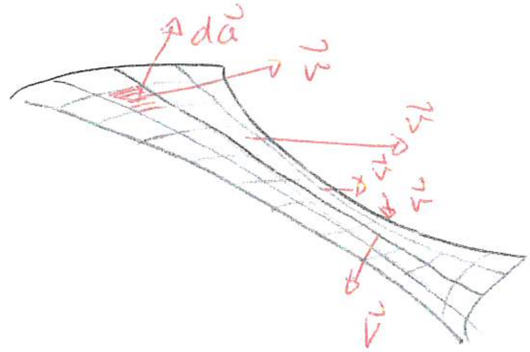
We define the area vector as

$$d\vec{a} = da\,\hat{n}$$

where $\hat{n}$ is the outward normal to the surface. Then, for this portion the volume flow rate is:

$$\vec{v} \cdot d\vec{a}$$

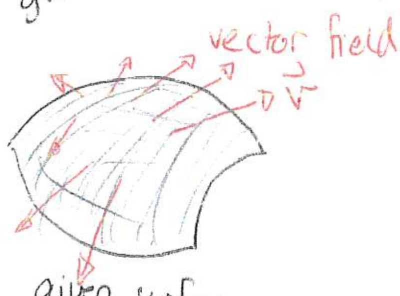
For a non-uniform velocity vector field and/or a curved surface we can break the surface into segments. The total flow rate is



$$\frac{dV}{dt} = \sum_{\substack{\text{all segments} \\ \text{as } da \to 0}} \vec{v} \cdot d\vec{a} \equiv \int_{\text{surface}} \vec{v} \cdot d\vec{a}$$

This is the idea behind a surface integral. It can be defined more precisely (see following pages)

given vector field



given surface

surface integral

$$\int \vec{v} \cdot d\vec{a}$$

is a scalar

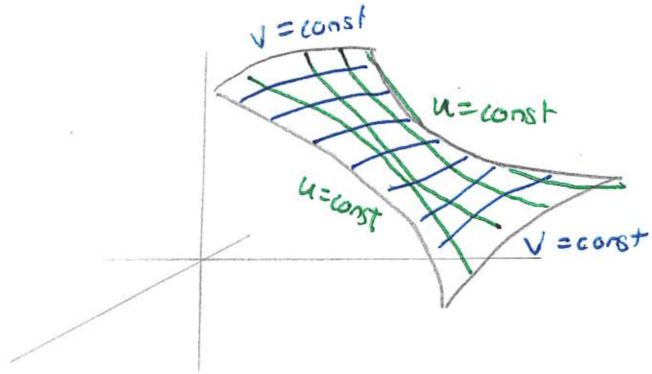
General surface integrals

Given a vector field, we construct a surface of integration by:

1) parameterizing the surface
2) constructing area vectors
3) constructing integrals

One can parameterize any two dimensional surface in $\mathbb{R}^3$ with two scalar parameters u, v. Then the surface is described by

$x(u,v)$, $y(u,v)$, $z(u,v)$



Then consider a vector field

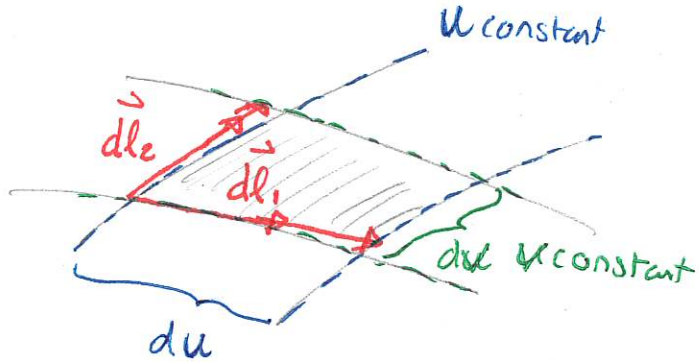
$$\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$$

This becomes a field that depends on the parameters on the surface

$$\vec{F} = \underbrace{F_x(x(u,v), y(u,v), z(u,v))}_{\text{function of } u,v \text{ on surface}} \hat{x} + \underbrace{F_y(x(u,v), y(u,v), z(u,v))} \hat{y} + \dots$$

Area vectors can be constructed from the u,v grid on the surface by

1) identify tangent vectors to the u,v grid



2) use the tangent vectors to construct a vector with
   * magnitude approximately the shaded area
   * direction perpendicular to the surface

The two illustrated vectors will do this. The shaded area is approximately

$$|\vec{dl}_1 \times \vec{dl}_2|$$

The perpendicular direction is $\vec{dl}_1 \times \vec{dl}_2$. Then

$$\vec{dl}_1 = \left[\frac{\partial x}{\partial u}\hat{x} + \frac{\partial y}{\partial u}\hat{y} + \frac{\partial z}{\partial u}\hat{z}\right] du$$

$$\vec{dl}_2 = \left[\frac{\partial x}{\partial v}\hat{x} + \frac{\partial y}{\partial v}\hat{y} + \frac{\partial z}{\partial v}\hat{z}\right] dv$$

Thus we define the area vector for the segment:

$$d\vec{a} = \vec{dl}_1 \times \vec{dl}_2$$

$$= \left[\frac{\partial x}{\partial u}\hat{x} + \frac{\partial y}{\partial u}\hat{y} + \frac{\partial z}{\partial u}\hat{z}\right] \times \left[\frac{\partial x}{\partial v}\hat{x} + \frac{\partial y}{\partial v}\hat{y} + \frac{\partial z}{\partial v}\hat{z}\right] du\,dv$$

$$= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) du\,dv\,\hat{x} + \dots$$

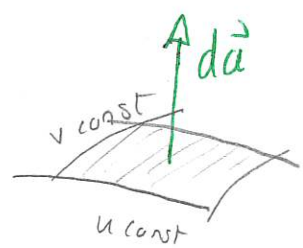
Then the surface integral is defined as:

$$\int_{\text{surface}} \vec{F}\cdot d\vec{a} = \int \vec{F}\cdot \vec{dl}_1 \times \vec{dl}_2$$

$$\Rightarrow \boxed{\int_{\text{surface}} \vec{F}\cdot d\vec{a} = \iint_{u\,v\text{ for surface}} du\,dv \left[F_x\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + F_y\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) + F_z\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\right]}$$

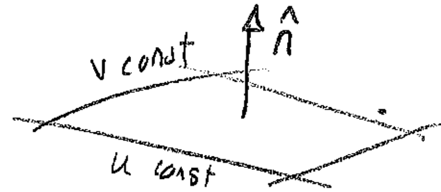
This is a two dimensional integral.

Note that there is a directional sense to the area vector and switching the parameters will reverse the direction of $d\vec{a}$ and thus the sign of the integral



We can bypass some of the construction details by defining a unit vector perpendicular to the surface, $\hat{n}$.

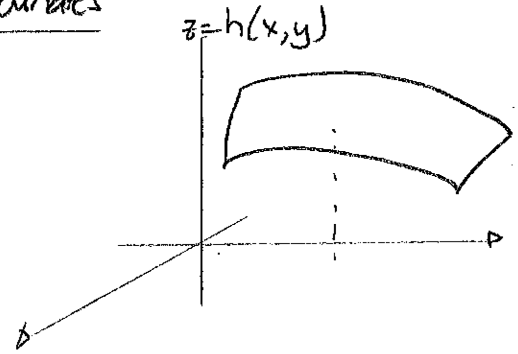
Then



$$d\vec{a} = \hat{n}\,du\,dv$$

and we can evaluate $\vec{F}\cdot d\vec{a}$ using this.

<u>Surface described in terms of Cartesian co-ordinates</u>



If the surface projects onto the x-y plane in a 1-1 fashion then we can parameterize in terms of x, y. So

$u = x \qquad z = h(x,y)$

$v = y$

Then

$$d\vec{a} = \left(\frac{\partial x}{\partial u}\hat{x} + \frac{\partial y}{\partial u}\hat{y} + \frac{\partial z}{\partial u}\hat{z}\right) \times \left(\frac{\partial x}{\partial v}\hat{x} + \frac{\partial y}{\partial v}\hat{y} + \frac{\partial z}{\partial v}\hat{z}\right) dx\,dy$$

$$= \Big(\underbrace{\frac{\partial x}{\partial x}}_{1}\hat{x} + \underbrace{\frac{\partial y}{\partial x}}_{0}\hat{y} + \frac{\partial h}{\partial x}\hat{z}\Big) \times \Big(\underbrace{\frac{\partial x}{\partial y}}_{0}\hat{x} + \underbrace{\frac{\partial y}{\partial y}}_{1}\hat{y} + \frac{\partial h}{\partial y}\hat{z}\Big)\, dx\,dy$$

$$= \left[\left(\hat{x} + \frac{\partial h}{\partial x}\hat{z}\right) \times \left(\hat{y} + \frac{\partial h}{\partial y}\hat{z}\right)\right] dx\,dy = \left[\hat{z} + \frac{\partial h}{\partial y}(-\hat{y}) - \frac{\partial h}{\partial x}\hat{x}\right] dx\,dy$$

$$= \left[-\frac{\partial h}{\partial x}\hat{x} - \frac{\partial h}{\partial y}\hat{y} + \hat{z}\right] dx\,dy$$

<u>Example</u>: For a hemisphere of radius r, $\quad z = h(x,y) = \sqrt{r^2 - x^2 - y^2}$

$$\frac{\partial h}{\partial x} = \frac{-x}{\sqrt{r^2-x^2-y^2}} \qquad \frac{\partial h}{\partial y} = -\frac{y}{\sqrt{r^2-x^2-y^2}}$$

$$\Rightarrow \quad d\vec{a} = \frac{1}{\sqrt{r^2-x^2-y^2}}\Big[x\hat{x} + y\hat{y} + \underbrace{\sqrt{}}_{z}\,\hat{z}\Big] \Rightarrow d\vec{a} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{r^2-x^2-y^2}}$$

This points radially outwards

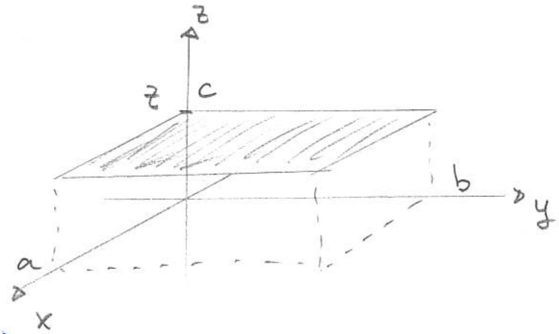
2 Surface integrals

a) Let $\mathbf{v} = z^2 x\hat{\mathbf{x}} + x^2 z\hat{\mathbf{z}}$. Determine the surface integral of $\mathbf{v}$ over the surface $0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b$ and $z = c$.

b) Let $\mathbf{v} = yx^2\hat{\mathbf{x}} + xy^2\hat{\mathbf{y}}$. Determine the surface integral of $\mathbf{v}$ over the flat surface in the region $0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b$ that slants from $z = c$ (at $y = 0$) to $z = 0$ (at $y = b$).

Answer: a) First list variables

$$\left.\begin{matrix} 0 \le x \le a \\ 0 \le y \le b \end{matrix}\right\}$$
$$z = c$$



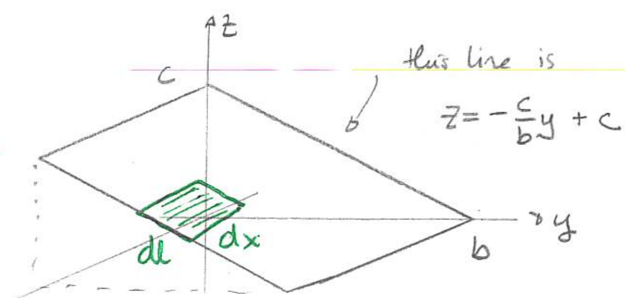
We can parameterize using x, y.
So

$$d\vec{a} = \underbrace{dx\,dy}_{\text{from the variables non-constant}}\ \hat{z} \leftarrow \text{from const variable}$$

Then $\vec{v}\cdot d\vec{a} = x^2 z\,dx\,dy = x^2 c\,dx\,dy$ since $z = c$

gives
$$\int \vec{v}\cdot d\vec{a} = \int_0^a dx \int_0^b dy\, x^2 c = c \underbrace{\int_0^a x^2 dx}_{a^3/3} \underbrace{\int_0^b dy}_{b}$$

$$\Rightarrow \int \vec{v}\cdot d\vec{a} = \frac{a^3 bc}{3}$$



b) List variables on surface

$0 \le x \le a$

$0 \le y \le b$

$$z = -\frac{c}{b}y + c = h(x,y)$$

We can construct the area vector using

$$d\vec{a} = \left[-\frac{\partial h}{\partial x}\hat{x} - \frac{\partial h}{\partial y}\hat{y} + \hat{z}\right] dx\,dy \Rightarrow d\vec{a} = \left[\frac{c}{b}\hat{y} + \hat{z}\right] dx\,dy$$

Then

$$\vec{v}\cdot d\vec{a} = xy^2 \frac{c}{b}\, dx\,dy$$

and

$$\int_{\text{surface}} \vec{v}\cdot d\vec{a} = \int_0^a dx \int_0^b dy\; xy^2 \frac{c}{b}$$

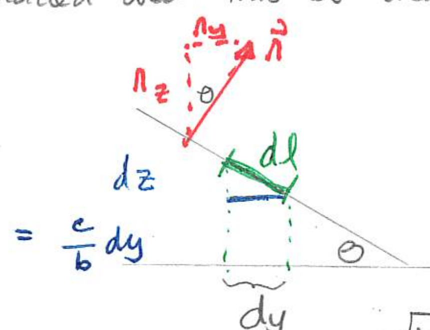
$$= \frac{c}{b} \underbrace{\int_0^a x\,dx}_{a^2/2} \underbrace{\int_0^b y^2\,dy}_{b^3/3} \qquad \Rightarrow \int \vec{v}\cdot d\vec{a} = \frac{a^2 b^2 c}{6}$$

Geometric construction of $d\vec{a}$

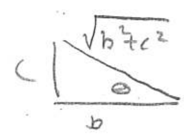
Alternatively we can construct $d\vec{a}$ via the shaded area. This has area

$$dl\,dx = \sqrt{dy^2 + \frac{c^2}{b^2}dy^2}\; dx$$

$$= \sqrt{1 + c^2/b^2}\; dx\,dy$$



Now we need the normal vector



$$\hat{n} = n_y \hat{y} + n_z \hat{z}$$

$$= n \sin\theta\, \hat{y} + n\cos\theta\, \hat{z} = \sin\theta\, \hat{y} + \cos\theta\, \hat{z} \qquad \text{since } n = 1.$$

But $\cos\theta = \frac{b}{\sqrt{b^2+c^2}} = \frac{1}{\sqrt{1+c^2/b^2}}$

$$\sin\theta = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{b}\frac{1}{\sqrt{1+c^2/b^2}}$$

$$\Rightarrow \hat{n} = \frac{1}{\sqrt{1+c^2/b^2}}\left[\frac{c}{b}\hat{y} + \hat{z}\right]$$

$$\Rightarrow d\vec{a} = n\,dl\,dx = \left[\frac{c}{b}\hat{y} + \hat{z}\right] dx\,dy$$

The rest of the integral follows as before