Phys 311 2025

Lecture 4

Fri: MW3 by 5pm

Tues: MW4 by 5pm

Corrections: HW1 by ~~Thurs~~ Fri.

HW 2 by Tues

Read: 1.3.3, 1.3.1 surface int.

Integration in three dimensions

We will encounter integrals of the following types:

1) integrals of scalar functions
2) integrals of vector functions — line and surface integrals.

We frequently connect integrals and derivatives. This is based on the fundamental theorem of calculus

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a).$$

Volume integrals of scalar functions

A common example of a scalar function in electromagnetic theory is the charge density. This describes the spatial arrangement of charge. Specifically:

The volume charge density $\rho(x,y,z)$ is a function with units C/m^3 and has the approximate meaning: The charge in the region

$x \to x+dx$

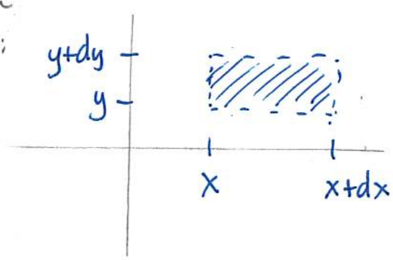
$y \to y+dy$

$z \to z+dz$

is

$$\underbrace{\rho(x,y,z)}_{\text{density}}\,\overbrace{dx\,dy\,dz}^{\text{volume}}$$

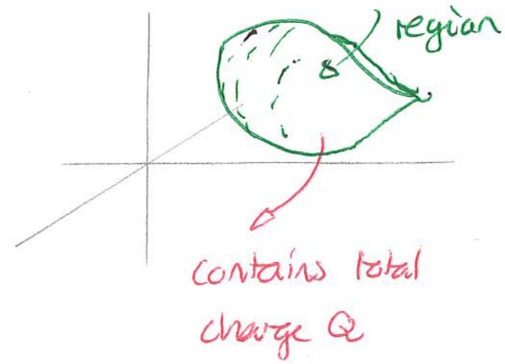
as $dx, dy, dz \to 0$



More precisely

Consider any region of space. Then the volume charge density is the function $\rho(x,y,z)$ with units C/m^3 so that the charge contained in the region is:

$$Q = \iiint_{\text{region}} \rho(x,y,z)\, dx\, dy\, dz$$



We often abbreviate the notation by using

$$dx\,dy\,dz \rightsquigarrow d\tau$$

and

$$Q = \int \rho\, d\tau.$$

This is deceptive

$$Q = \int \rho(x,y,z)\, d\tau$$

Does NOT mean integrate with respect to one variable τ.

Does Mean integrate with respect to x, y, z (three variables)

Is NOT usually constant. CANNOT usually be pulled out of the integral

1 Three dimensional charge distribution

Within the region $-a \leqslant x, y, z \leqslant a$ with $a > 0$ the charge density is

$$\rho(x, y, z) = \frac{4q}{a^5} xy$$

where q is a constant with dimensions of charge.

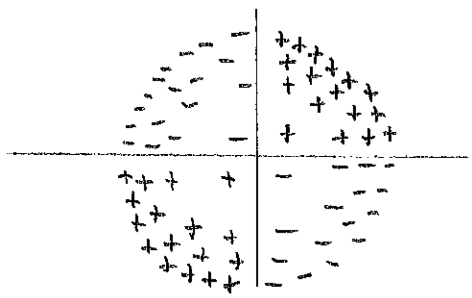
a) In the region $-a \leqslant xy \leqslant a$ in the xy plane sketch whether the charge is positive or negative indicating regions with greater and smaller charge density.

b) Determine the total charge in the region $-a \leqslant x, y, z \leqslant a$.

c) Determine the total charge in the region $0 \leqslant x, y, z \leqslant a$.

d) Determine the total charge contained in the segment of a sphere of radius a in the quadrant where $0 \leqslant x, y, z \leqslant a$.

Answer: a)



b) the region has

$$\left.\begin{array}{l} -a \leq x \leq a \\ -a \leq y \leq a \\ -a \leq z \leq a \end{array}\right\} d\tau = dx\,dy\,dz$$

$$Q = \int \rho \, d\tau = \int_{-a}^{a} dz \int_{-a}^{a} dy \int_{-a}^{a} dx \, \frac{4q}{a^5} xy$$

$$= \frac{4q}{a^5} \underbrace{\int_{-a}^{a} dz}_{2a} \underbrace{\int_{-a}^{a} y\,dy}_{\underbrace{\left.\frac{y^2}{2}\right|_{-a}^{a}}_{=0}} \underbrace{\int_{-a}^{a} x\,dx}_{\underbrace{\left.\frac{x^2}{2}\right|_{-a}^{a}}_{=0}}$$

$$\Rightarrow \quad Q = 0$$

c) $\left.\begin{array}{l} 0 \le x \le a \\ 0 \le y \le a \\ 0 \le z \le a \end{array}\right\}$ $d\tau = dx\,dy\,dz$

$$Q = \int \rho \, d\tau = \int_0^a dz \int_0^a dy \int_0^a dx \; \frac{4q}{a^5} xy$$

$$\Rightarrow Q = \frac{4q}{a^5} \int_0^a dz \int_0^a y\,dy \int_0^a x\,dx$$

$$= \frac{4q}{a^5} \, a \, \frac{y^2}{2}\Big|_0^a \, \frac{x^2}{2}\Big|_0^a$$

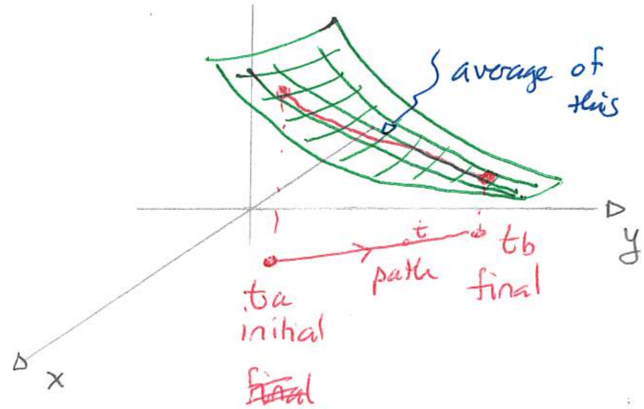
$$= \frac{4q}{a^5} \, a \, \frac{a^2}{2} \, \frac{a^2}{2}$$

$$\Rightarrow Q = q.$$

Line integrals

Line integrals consider situations where one wants to aggregate information of a function of multiple variables but only along a line through space.

For example suppose that one traversed a landscape, represented by position co-ordinates x and y and altitude $h(x,y)$. To do this requires:



1) the function $h(x,y)$
2) a description of the trajectory - a path in x y space
   - parameterized by $t_a \leq t \leq t_b$
   - $x(t), y(t)$

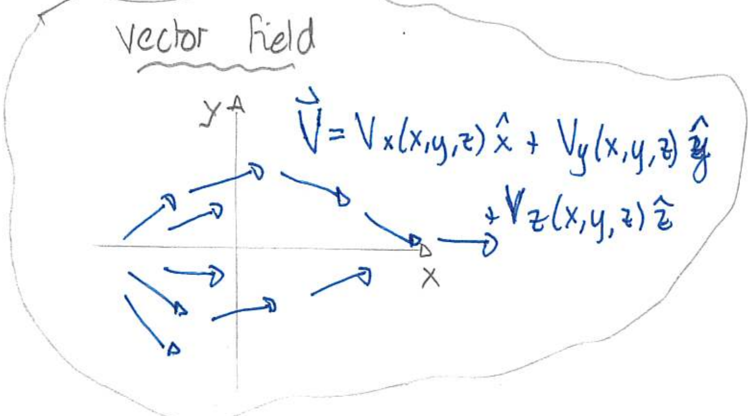
Then at the location represented by t,

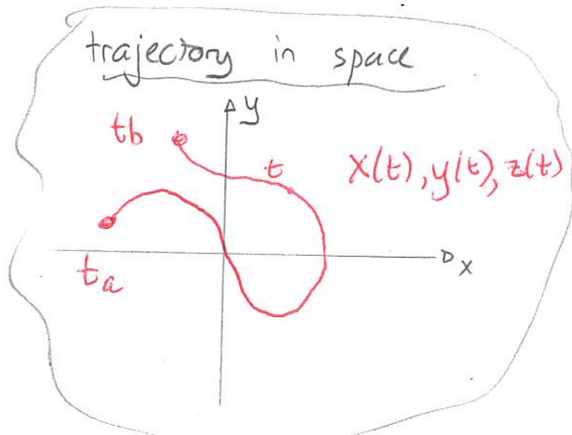
$$\text{altitude at } t = h(x(t), y(t))$$

This is now a function of one variable and we could say that the average height is

$$\bar{h} = \frac{1}{\underset{(t_b - t_a)}{\text{total distance}}} \int_{t_a}^{t_b} h(x(t), y(t))\, dt$$

This is a line integral of a scalar function. Electromagnetic theory requires line integrals of vector functions. For these we need:



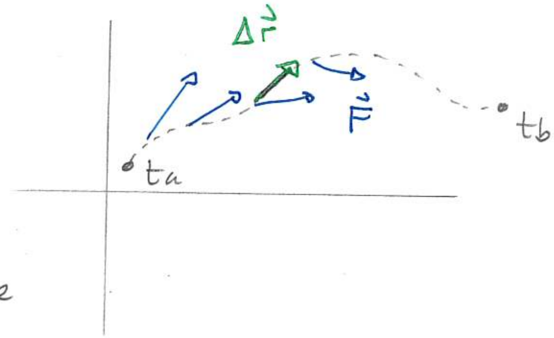


The are many possible constructions. In all cases we want the line integral to produce a result: that is independent of:

1) the Cartesian co-ordinate system in use
2) the parameterization of the curve.

One possibility, which will appear in electromagnetic theory, stems from mechanical work. Consider a force that depends on location $\vec{F} = \vec{F}(x,y,z)$ and suppose that a particle moves along a particular trajectory. We then decompose the trajectory into segments, each described by displacement $\Delta\vec{r}$. Then the work done by the force is

$$W = \sum_{\text{segments}} \vec{F} \cdot \Delta\vec{r}$$



Now $\Delta\vec{r} = \Delta x\,\hat{x} + \Delta y\,\hat{y} + \Delta z\,\hat{z}$ and if the path is parameterized by t so that the parameter changes by Δt along $\Delta\vec{r}$, we get

$$\Delta\vec{r} \cong \left[\frac{\Delta x}{\Delta t}\hat{x} + \frac{\Delta y}{\Delta t}\hat{y} + \frac{\Delta z}{\Delta t}\hat{z}\right]\Delta t$$

Thus

$$W \cong \sum \vec{F} \cdot \left[\frac{\Delta x}{\Delta t}\hat{x} + \frac{\Delta y}{\Delta t}\hat{y} + \frac{\Delta z}{\Delta t}\hat{z}\right]\Delta t$$

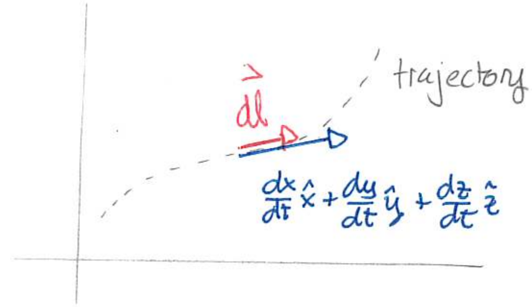
Taking $\Delta t \to 0$ gives

$$W = \int_{t_a}^{t_b} \vec{F} \cdot \left[\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right] dt$$

$$= \int_{t_a}^{t_b} \underbrace{\left[\underset{\uparrow\ \text{co-ord label}}{F_x}(x(t),y(t),z(t))\frac{dx}{dt} + F_y(x(t),y(t),z(t))\frac{dy}{dt} + F_z(x(t),y(t),z(t))\frac{dz}{dt}\right]}_{\text{Function of } t \text{ constructed from } \vec{F} \text{ and trajectory}} dt$$

Note that

$$\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$



is a vector that is tangent to the trajectory.

It follows that

$$\vec{dl} = \left[\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right] dt = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

is also tangent to the trajectory. So we might express the work as

$$W = \int_{\text{trajectory}} \vec{F}\cdot\vec{dl}$$

In general we define:

The line integral of

$$\vec{V} = V_x(x,y,z)\,\hat{x} + V_y(x,y,z)\,\hat{y} + V_z(x,y,z)\,\hat{z}$$

along the curve described by

$$\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$$

is

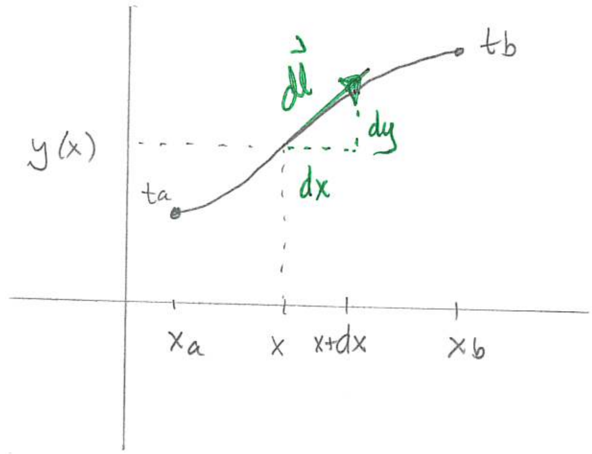
$$\int \vec{V}\cdot\vec{dl} = \int_{t_a}^{t_b} \left[V_x(x(t),y(t),z(t))\frac{dx}{dt} + V_y(\dots)\frac{dy}{dt} + V_z(\dots)\frac{dz}{dt}\right] dt$$

where $t_a \le t \le t_b$ is the range of parameters along the curve segment.

We can show:

1) this is independent of the parametrization of the curve
2) this is independent of the co-ordinate system or the basis vectors (on which V_x, V_y and V_z depend)
3) It could depend on the path

In some situations we can use the co-ordinates as integration variables. For example if the path is such that each value of x corresponds to one value of y then the curve can be parameterized by x. So



$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$= dx\,\hat{x} + \frac{dy}{dx}dx\,\hat{y} + \frac{dz}{dx}dz\,\hat{z}$$

$$= \left[\hat{x} + \frac{dy}{dx}\hat{y} + \frac{dz}{dx}\hat{z}\right]dx$$

So

$$\int \vec{v}\cdot\vec{dl} = \int_{x_a}^{x_b} \left[v_x + v_y\frac{dy}{dx} + v_z\frac{dz}{dx}\right]dx$$

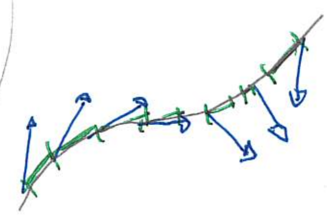
Qualitatively

The line integral aggregates the component of

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

along the tangent to the trajectory

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

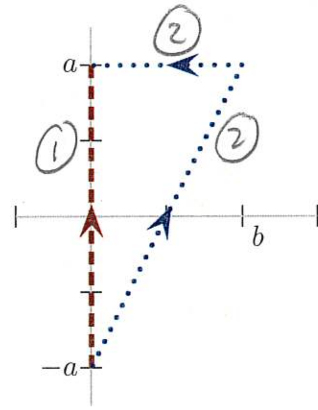


2 Line integral in two dimensions: straight segments

Let $\mathbf{v} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$. Determine

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

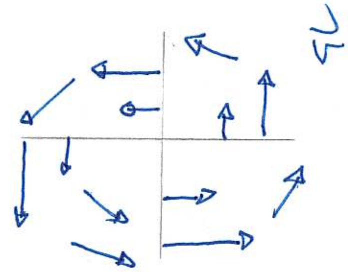
along each of the two paths.



The vector field has components

$$V_x(x,y,z) = -y$$
$$V_y(x,y,z) = x$$
$$V_z(x,y,z) = 0$$

A sketch indicates that $\vec{v}$ circles



Along path 1

We can parameterize the path using y. Then

$$-a \leq y \leq a$$
$$x = 0$$

So

$$\int \vec{v} \cdot \vec{dl} = \int V_x\, dx + V_y\, dy$$

and $dx = \frac{dx}{dy} dy = 0$ gives

$$\vec{v} \cdot \vec{dl} = V_x\, dx + V_y\, dy = \Big[V_x \cancel{\frac{dx}{dy}}^{0} + V_y\Big] dy$$

Along this path $V_y = x = 0$. Thus

$$\vec{v} \cdot \vec{dl} = 0\, dy$$

$$\Rightarrow \int \vec{v} \cdot \vec{dl} = \int 0\, dy = 0 \quad \Rightarrow \boxed{\int \vec{v} \cdot \vec{dl} = 0}$$

Matches the sketch since $\vec{v} \perp \vec{dl}$ everywhere along this path

Along path ② . It appears that $\vec{v}\cdot d\vec{l} > 0$ everywhere so the integral should be positive. We break it into

$$\int \vec{v}\cdot d\vec{l} = \int_{\text{diagonal}} \vec{v}\cdot d\vec{l} + \int_{\text{horizontal.}} \vec{v}\cdot d\vec{l}$$

Along the diagonal $\quad 0 \leq x \leq b \quad$ so we use x as the parameter

$$y = \frac{2a}{b}x - a$$

$$\vec{v}\cdot d\vec{l} = v_x dx + v_y dy = v_x dx + v_y \frac{dy}{dx}dx = \left[v_x + v_y \frac{dy}{dx}\right]dx$$

Then $\quad v_x = -y = \frac{-2a}{b}x + a$

$$v_y = x$$

$$\frac{dy}{dx} = \frac{2a}{b}.$$

Thus

$$\vec{v}\cdot d\vec{l} = \left[-\frac{2a}{b}x + a + x\frac{2a}{b}\right]dx = a\,dx$$

$$\Rightarrow \int_{\text{diagonal}} \vec{v}\cdot d\vec{l} = \int_0^b a\,dx = ab. \Rightarrow \int_{\text{diagonal}} \vec{v}\cdot d\vec{l} = ab.$$

Along the horizontal

$$\int \vec{v}\cdot d\vec{l}\ (\longleftarrow) = -\int \vec{v}\cdot d\vec{l}\ (\longrightarrow)$$

$$\Rightarrow \int_{\text{right to left}} \vec{v}\cdot d\vec{l} = -\int_{\text{left to right}} \vec{v}\cdot d\vec{l}$$

For the left to right integral

$0 \leq x \leq b \leftarrow$ x is parameter

$y = a \Rightarrow \frac{dy}{dx} = 0$

$$\vec{v}\cdot d\vec{l} = v_x dx + v_y \underbrace{\frac{dy}{dx}}_{0} dx$$

$$= -y\,dx + 0 = -a\,dx$$

$$\int_{l \text{ to } r} \vec{v}\cdot d\vec{l} = \int_0^b -a\,dx = -ab.$$

Thus $\int_{r \text{ to } l} \vec{v}\cdot d\vec{l} = ab$

$$\Rightarrow \int_{\text{path 2}} \vec{v}\cdot d\vec{l} = 2ab$$

The example illustrates the fact that

Line integrals generally depend on the path between initial + final locations