Tues: HW 2 by 5pm

Fri: HW 3 by 5pm

HW 1:

Thurs: Seminar 12:30pm W.S 160

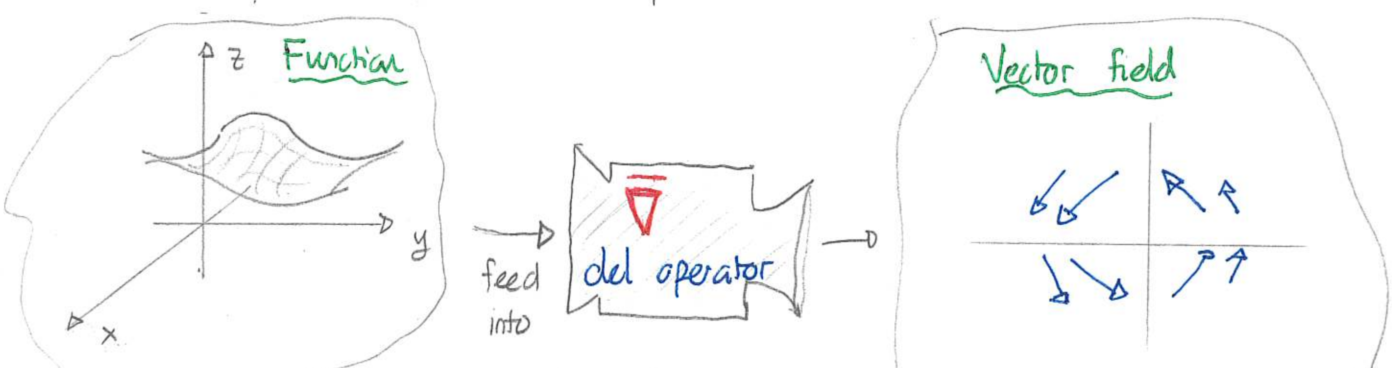
Thurs: Read

Differentiation in Three Dimensions

The gradient describes a three-dimensional analog of a derivative. Recall that for a function $f(x,y,z)$, the gradient is

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

We then consider a mathematical operation that does this.



e.g. $f = \alpha \frac{y}{x}$

$\vec{\nabla} f = -\alpha \frac{y}{x^2}\hat{x} + \alpha \frac{1}{x}\hat{y}$

We can think of the operator as having an existence independent of its input and output. We write

$$f(x,y,z) \xrightarrow{\vec{\nabla}} \vec{\nabla} f$$

or

$$\vec{\nabla}(\ldots) = \hat{x}\frac{\partial}{\partial x}(\ldots) + \hat{y}\frac{\partial}{\partial y}(\ldots) + \hat{z}\frac{\partial}{\partial z}(\ldots)$$

insert function

Then we write

$$\boxed{\vec{\nabla} := \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$$

Notes:

1) $\vec{\nabla}$ is not a function in the usual sense. A function takes numbers as input and $\vec{\nabla}$ takes a function as an input.

2) The derivatives do not act on basis vectors, which can be regarded as independent variables. So

$$\frac{\partial}{\partial x} x = 1 \quad \text{and} \quad \frac{\partial}{\partial x}\hat{x} \neq 1$$

3) Remember to include basis vectors when using $\vec{\nabla}$. So

$$\vec{\nabla} f = \underline{\hat{x}}\frac{\partial f}{\partial x} + \underline{\hat{y}}\frac{\partial f}{\partial y} + \underline{\hat{z}}\frac{\partial f}{\partial z} \quad \text{vectors}$$

$$\neq \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \quad \text{scalar.}$$

Calculus on vector fields

Are there ways to differentiate a vector field? Any type of differentiation of a vector field

1) will compare the fields at nearby locations and find changes in the fields
2) must give the same results under different orthogonal co-ordinate systems.

In general a vector field can be expressed as:

$$\vec{V} = \vec{V}\underbrace{(x,y,z)}_{\text{describes location}} = V_{\underbrace{x}_{\text{labels component}}}\underbrace{(x,y,z)}_{\text{location}}\underbrace{\hat{x}}_{\text{basis vector}} + V_y(x,y,z)\hat{y} + V_z(x,y,z)\hat{z}$$

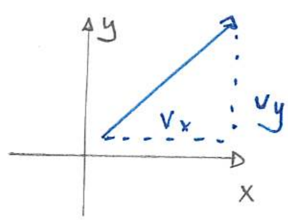
We will provide an example of a derivative that does NOT give the same outcome in all orthogonal basis

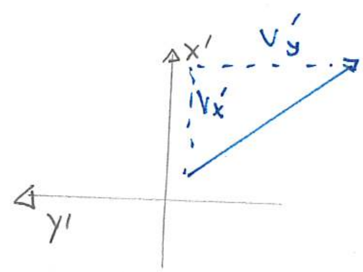
Example (Bad definition)

Suppose we define

$$\frac{\partial V_x}{\partial x}\hat{x} + \frac{\partial V_y}{\partial y}\hat{y} + \frac{\partial V_z}{\partial z}\hat{z}$$

where x, y, z is a Cartesian co-ordinate system. Consider this for the following co-ordinates





Then we can see that any single vector $\vec{V}$ can be written as

$$\vec{V} = V_x\hat{x} + V_y\hat{y} \quad \text{or} \quad \vec{V} = V_x'\hat{x}' + V_y'\hat{y}'$$

But $\hat{x}' = \hat{y}$ and $\hat{y}' = -\hat{x}$ gives

$$\vec{V} = V_x'\hat{x}' + V_y'\hat{y}' = V_x'\hat{y} - V_y'\hat{x} = V_x\hat{x} + V_y\hat{y}$$

$$\Rightarrow \quad \begin{matrix} V_x = -V_y' \\ V_y = V_x' \end{matrix} \quad \text{or} \quad \begin{matrix} V_x' = V_y \\ V_y' = -V_x \end{matrix}$$

We must use the same procedure in each frame, and then translate results using

$$V_x' = V_y \qquad x = -y' \qquad \hat{x}' = \hat{y}$$
$$V_y' = -V_x \qquad y = x' \qquad \hat{y}' = -\hat{x}$$

Suppose there are no z-components.

Unprimed system

$$\frac{\partial V_x}{\partial x}\hat{x} + \frac{\partial V_y}{\partial y}\hat{y} + \frac{\partial V_z}{\partial z}\hat{z}$$

$$= \frac{\partial V_x}{\partial x}\hat{x} + \frac{\partial V_y}{\partial y}\hat{y}$$

Primed system

$$\frac{\partial V_x'}{\partial x'}\hat{x}' + \frac{\partial V_y'}{\partial y'}\hat{y}' + \dots$$

$$= \frac{\partial V_y}{\partial x'}\hat{y} + \frac{\partial(-V_x)}{\partial y'}(-\hat{x})$$

$$= \left[\frac{\partial V_y}{\partial x}\underbrace{\cancel{\frac{\partial x}{\partial x'}}}_{0} + \frac{\partial V_y}{\partial y}\underbrace{\frac{\partial y}{\partial x'}}_{1}\right]\hat{y} + \left[\frac{\partial V_x}{\partial x}\underbrace{\frac{\partial x}{\partial y'}}_{-1} + \frac{\partial V_x}{\partial y}\underbrace{\cancel{\frac{\partial y}{\partial y'}}}_{0}\right]\hat{x}$$

$$= \frac{\partial V_y}{\partial y}\hat{y} - \frac{\partial V_x}{\partial x}\hat{x}$$

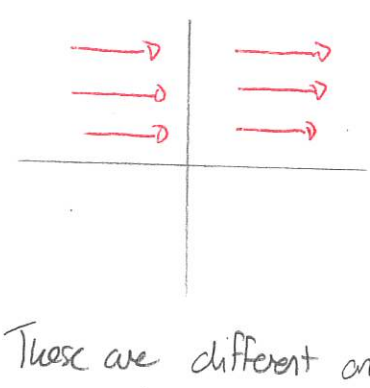
$$= -\frac{\partial V_x}{\partial x}\hat{x} + \frac{\partial V_y}{\partial y}\hat{y}$$

We can see that this definition will give different results in the two systems. Suppose for example, that $V = x\,\hat{x}$. Then $V_x = x$, $V_y = 0$

and

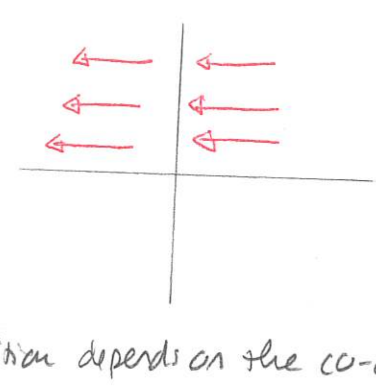
Unprimed

$\rightarrow \hat{x}$



Primed

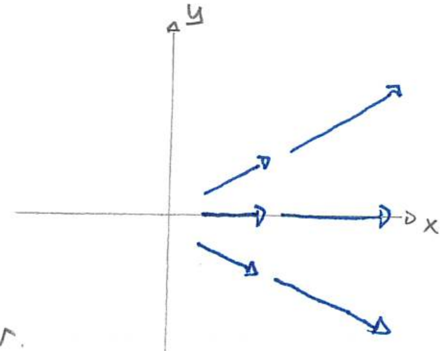
$-\hat{x}$



These are different and the definition depends on the co-ordinates. We do not want that.

Divergence of a vector field

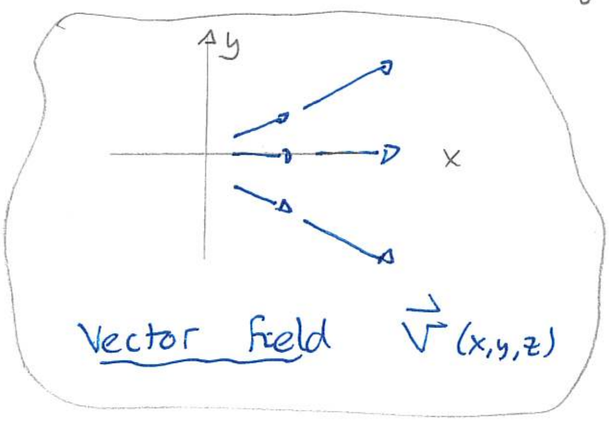
Consider the illustrated vector field. The field appears to diverge outwards and it seems that



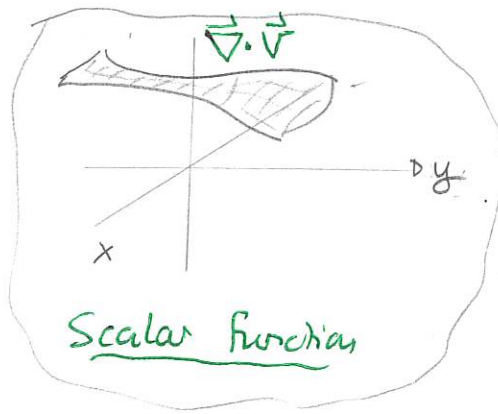
$$\frac{\partial V_x}{\partial x} > 0 \quad \text{and} \quad \frac{\partial V_y}{\partial y} > 0$$

We would like to capture this "diverging" behavior.

To do so we define the divergence of a vector field



divergence
$\longrightarrow$
operation



This is defined via:

$$\boxed{\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}}$$

LOOK!!
No vectors

Notess:

1) the divergence produces a scalar function; there are no vectors left
2) the divergence satisfies

$$\vec{\nabla}\cdot(\vec{u}+\vec{V}) = \vec{\nabla}\cdot\vec{u} + \vec{\nabla}\cdot\vec{V}$$

3) the divergence yields the same result regardless of the Cartesian co-ordinate system.
4) the divergence must be modified for non-Cartesian co-ordinate systems
5) $\vec{\nabla}\cdot\vec{V} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(V_x\hat{x} + V_y\hat{y} + V_z\hat{z}\right)$

$$= \hat{x}\frac{\partial}{\partial x}V_x\cdot\hat{x} + \hat{x}\frac{\partial}{\partial x}V_y\hat{y} + \dots + \hat{y}\frac{\partial}{\partial y}V_x\cdot\hat{x} + \hat{y}\frac{\partial}{\partial y}V_y\hat{y} + \dots$$

$$= \frac{\partial V_x}{\partial x}\overset{1}{\cancel{\hat{x}\cdot\hat{x}}} + \frac{\partial V_y}{\partial x}\overset{0}{\cancel{\hat{x}\cdot\hat{y}}} + \dots + \frac{\partial V_x}{\partial y}\underset{0}{\cancel{\hat{y}\cdot\hat{x}}} + \frac{\partial V_y}{\partial y}\underset{1}{\cancel{\hat{y}\cdot\hat{y}}} + \dots$$

1 Divergence

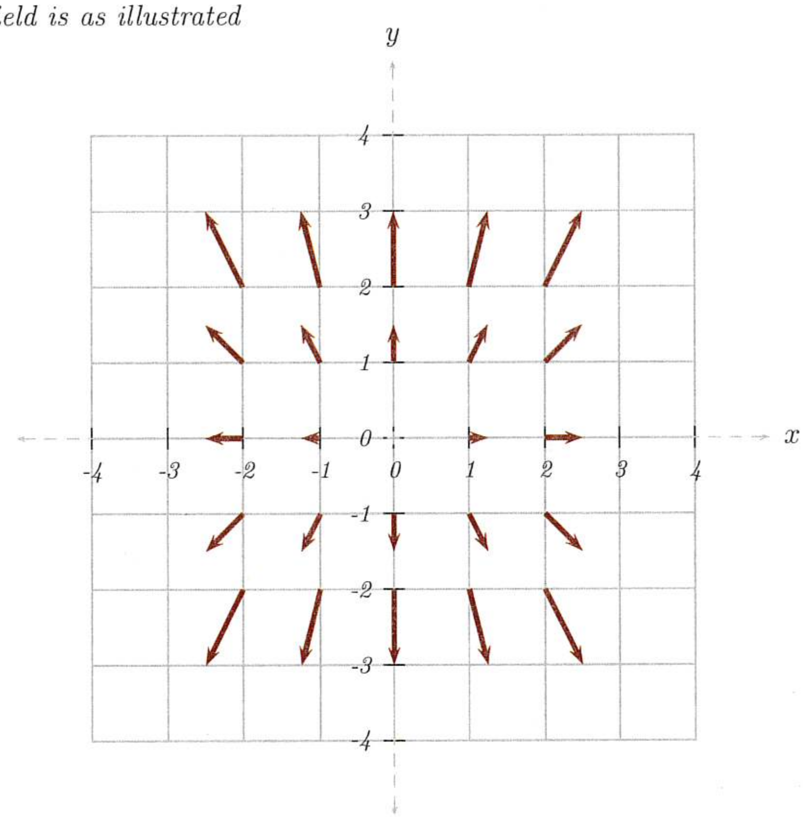
For each of the following, **sketch** the vector field and **determine its divergence.**

a) $\mathbf{v} = \dfrac{x}{4}\,\hat{\mathbf{x}} + \dfrac{y}{2}\,\hat{\mathbf{y}}$

b) $\mathbf{v} = \dfrac{-y}{2}\,\hat{\mathbf{x}} + \dfrac{x}{2}\,\hat{\mathbf{y}}$

Answer:

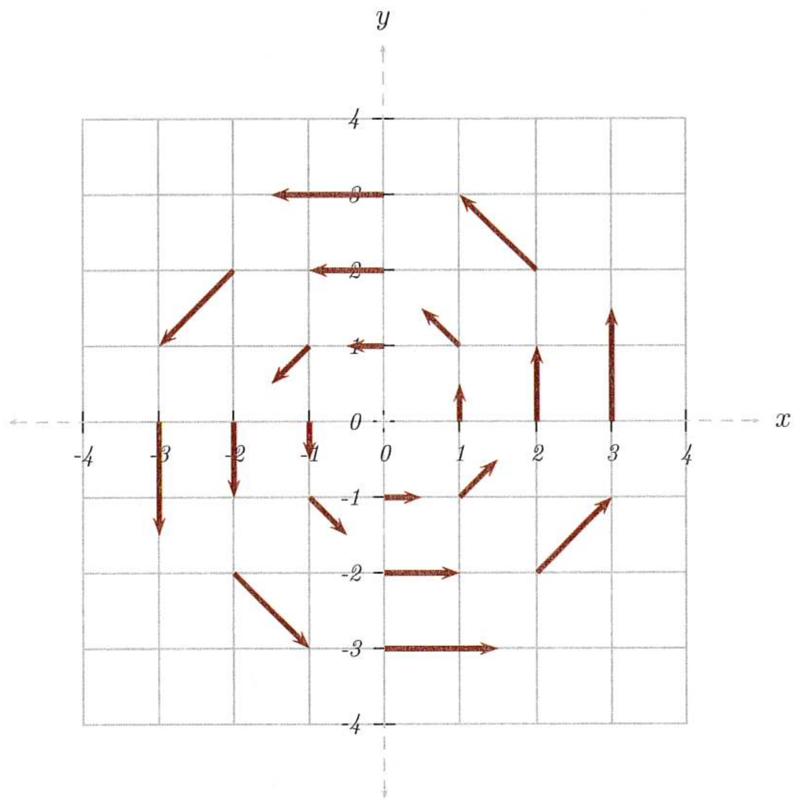
a) The vector field is as illustrated



The components are $v_x = x/4$ *and* $v_y = y/2$. *The divergence is*

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\ &= \frac{\partial}{\partial x}\left(\frac{x}{4}\right) + \frac{\partial}{\partial y}\left(\frac{y}{2}\right) \\ &= \frac{3}{4}.\end{aligned}$$

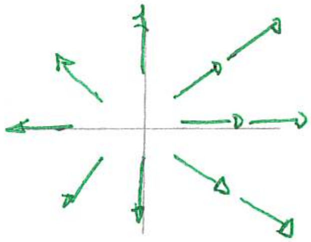
b) The vector field is as illustrated



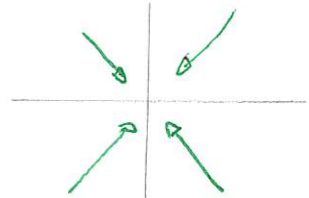
The components are $v_x = -y/2$ *and* $v_y = x/2$*. The divergence is*

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\ &= \frac{\partial}{\partial x}\left(-\frac{y}{2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{2}\right) \\ &= 0.\end{aligned}$$

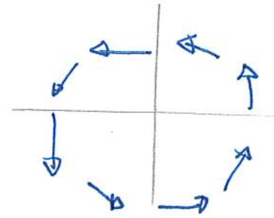
It is typical to find the following, although there are exceptions.



$\vec{\nabla}\cdot\vec{V} > 0$



$\vec{\nabla}\cdot\vec{V} < 0$



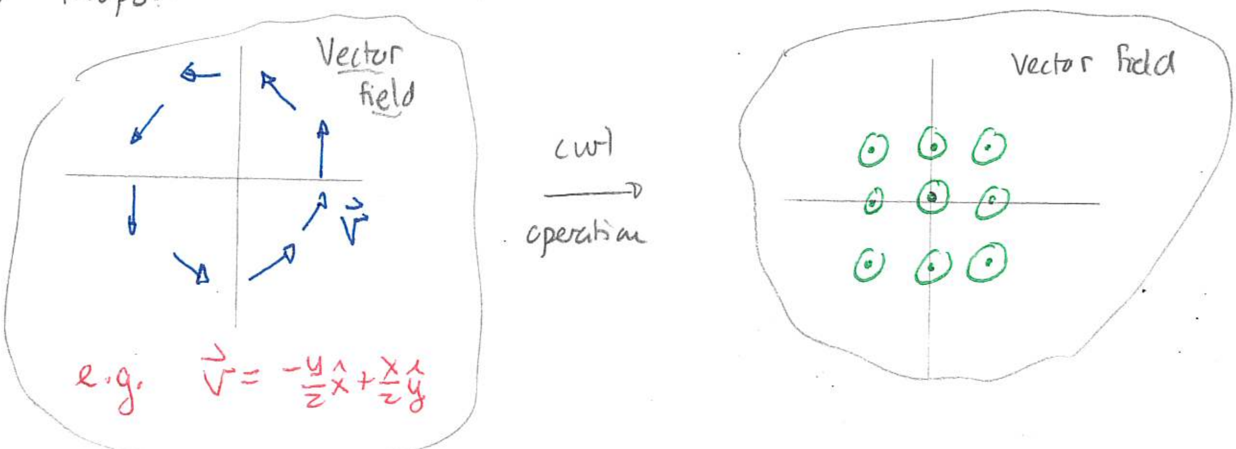
$\vec{\nabla}\cdot\vec{V} = 0$

Curl of a vector field

Some vector fields have a rotational aspect. The curl captures this and is defined via:

> Suppose that, in Cartesian co-ordinates
>
> $$\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$$
>
> Then the curl of $\vec{V}$ is
>
> $$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{x} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{y} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z}$$

This maps:



Note:

1) the curl of a vector field does not depend on the choice of Cartesian basis

2) the curl is a vector and the output must include unit vectors.

3) $\vec{\nabla} \times (\vec{u} + \vec{v}) = \vec{\nabla} \times \vec{u} + \vec{\nabla} \times \vec{v}$

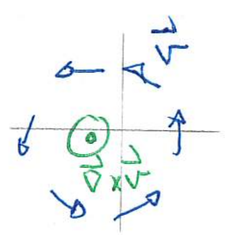
4) $\vec{\nabla} \times \vec{v} = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \times \left(V_x \hat{x} + V_y \hat{y} + V_z \hat{z} \right)$

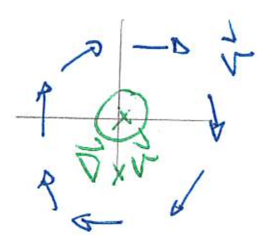
$$= \hat{x} \frac{\partial}{\partial x} V_x \times \hat{x} + \hat{x} \frac{\partial}{\partial x} V_y \times \hat{y} + \ldots$$

$$= \frac{\partial V_x}{\partial x} \underbrace{\cancel{\hat{x} \times \hat{x}}}_{0} + \frac{\partial V_y}{\partial x} \hat{x} \times \hat{y} + \ldots$$

$$= \frac{\partial V_y}{\partial x} \hat{z} + \ldots$$

5) qualitatively a right-hand rule gives the direction of the curl.





2 Curl examples

Determine the curl of:

a) $\mathbf{v} = \frac{x}{4}\,\hat{\mathbf{x}} + \frac{y}{2}\,\hat{\mathbf{y}}$

b) $\mathbf{v} = \frac{-y}{2}\,\hat{\mathbf{x}} + \frac{x}{2}\,\hat{\mathbf{y}}$

Answer:

a) Here

$$\begin{aligned} v_x &= \frac{x}{4} \\ v_y &= \frac{y}{2} \end{aligned}$$

Thus

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{4} & \dfrac{y}{2} & 0 \end{vmatrix} \\ &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{y}{2}\right)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}\left(\frac{x}{4}\right) - \frac{\partial}{\partial x}(0)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{y}{2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{4}\right)\right] \\ &= \hat{\mathbf{x}}\,[0-0] + \hat{\mathbf{y}}\,[0-0] + \hat{\mathbf{z}}\,[0-0] \\ &= 0. \end{aligned}$$

b) Here

$$\begin{aligned} v_x &= -\frac{y}{2} \\ v_y &= \frac{x}{2} \end{aligned}$$

Thus

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{y}{2} & \dfrac{x}{2} & 0 \end{vmatrix} \\ &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{2}\right)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}\left(\frac{-y}{2}\right) - \frac{\partial}{\partial x}(0)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{2}\right)\right] \\ &= \hat{\mathbf{x}}\,[0-0] + \hat{\mathbf{y}}\,[0-0] + \hat{\mathbf{z}}\left[\frac{1}{2} + \frac{1}{2}\right] \\ &= \hat{\mathbf{z}}. \end{aligned}$$

Vector derivatives of sums and products 1.2.6

One can differentiate combinations of sums and products. For sums the derivatives are linear:

$$\boxed{\begin{aligned} &\text{gradient:} && \vec{\nabla}(f+g) = \vec{\nabla} f + \vec{\nabla} g \\ &\text{divergence:} && \vec{\nabla}\cdot(\vec{u}+\vec{v}) = \vec{\nabla}\cdot\vec{u} + \vec{\nabla}\cdot\vec{v} \\ &\text{curl} \;: && \vec{\nabla}\times(\vec{u}+\vec{v}) = \vec{\nabla}\times\vec{u} + \vec{\nabla}\times\vec{v} \end{aligned}}$$

The rules involving multiplication can be more complicated but they are derived from the basic differentiation product rule:

1) gradient

$$\boxed{\begin{aligned} &\nabla(fg) = f\vec{\nabla} g + g\vec{\nabla} f \\ &\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} \end{aligned}}$$

Here $(\vec{A}\cdot\vec{\nabla})\vec{B} \neq (\vec{\nabla}\cdot\vec{A})\vec{B}$. Rather

$$(\vec{A}\cdot\vec{\nabla})\vec{B} = \left[(A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\right]\vec{B}$$

$$= \left[A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right]\left[B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\right]$$

$$(\vec{A}\cdot\vec{\nabla})\vec{B} = \left[A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right]\hat{x} + \left[A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right]\hat{y}$$
$$+ \left[A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right]\hat{z}$$

2) divergence

$$\vec{\nabla}\cdot(f\vec{A}) = (\vec{\nabla} f)\cdot\vec{A} + f\,\vec{\nabla}\cdot\vec{A}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = B\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

3) curl see pg 21 section 1.2.6

3 Divergence of a dot product

In general

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}$$

a) Verify that this is true for

$$\mathbf{A} = x^2 \hat{\mathbf{x}}$$
$$\mathbf{B} = \hat{\mathbf{x}}$$

b) Check that, for these vectors, $(\mathbf{B} \cdot \nabla) \mathbf{A} \neq (\nabla \cdot \mathbf{B}) \mathbf{A}$.

Answer: a) LHS $\vec{A} \cdot \vec{B} = x^2$ $\qquad \vec{\nabla}(\vec{A} \cdot \vec{B}) = \frac{\partial x^2}{\partial x} \hat{x} + \ldots = 2x \hat{x}$

RHS $\vec{\nabla} \times \vec{B} = 0$ since $\vec{B}$ is constant

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 0 & 0 \end{vmatrix} = \hat{x}\, 0 + \hat{y} \frac{\partial x^2}{\partial z} + \hat{z}\left(-\frac{\partial x^2}{\partial y}\right) = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{A} = 0$$

$$(\vec{A} \cdot \vec{\nabla}) \vec{B} = \left(A_x \frac{\partial}{\partial x} + \cancel{A_y \frac{\partial}{\partial y}} + \cancel{A_z \frac{\partial}{\partial z}} \right) \vec{B}$$

$$= x^2 \frac{\partial}{\partial x} \hat{x} = 0$$

$$(\vec{B} \cdot \vec{\nabla}) \vec{A} = \left(B_x \frac{\partial}{\partial x} + \cancel{B_y \frac{\partial}{\partial y}} + \cancel{B_z \frac{\partial}{\partial z}} \right) x^2 \hat{x} = \frac{\partial}{\partial x} x^2 \hat{x} = 2x \hat{x}$$

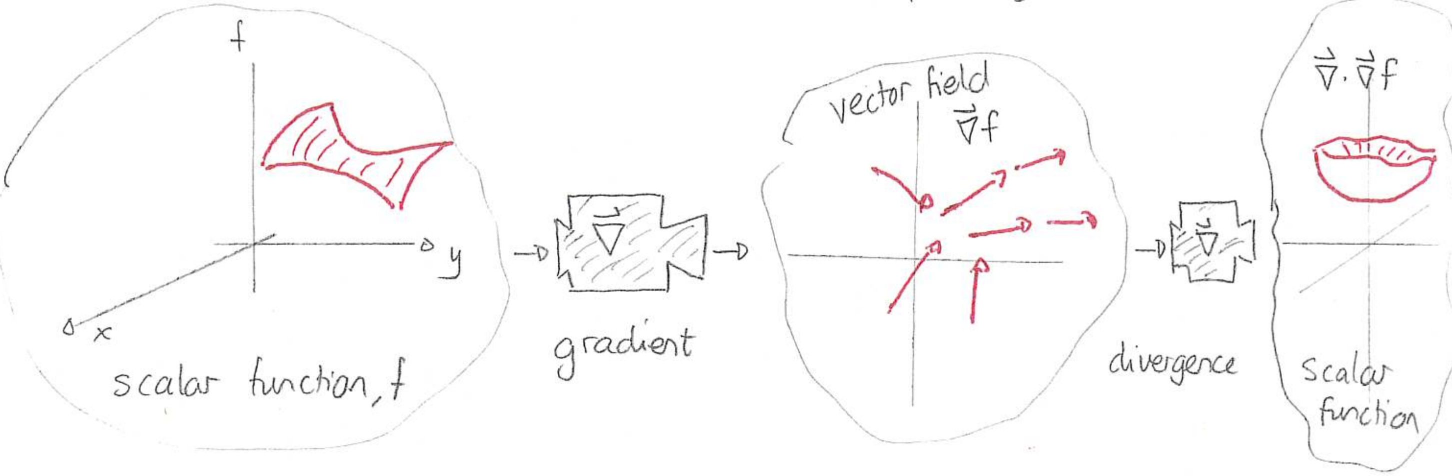
Adding all RHS terms gives $2x \hat{x}$ = LHS

b) $(\vec{B} \cdot \vec{\nabla}) \vec{A} = 2x \hat{x}$ $\qquad$ so $\quad (\vec{B} \cdot \vec{\nabla}) \vec{A} \neq (\vec{\nabla} \cdot \vec{B}) \vec{A}$

$(\underbrace{\vec{\nabla} \cdot \vec{B}}_{0}) \vec{A} = 0$

Multiple derivatives

Three dimensional differentiation can be done repeatedly. For example



In this example:

$$\vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla} \cdot \left[\frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} \right]$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is a common operation in electromagnetic theory (in the context of electric potential), other branches of physics and also mathematics. Thus we define

> The Laplacian operator, $\vec{\nabla}^2$, maps functions to functions via
>
> $$f \xrightarrow{\vec{\nabla}^2} \vec{\nabla}^2 f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We will eventually see that

$$\vec{\nabla}^2 V = \text{function of charge distribution}$$

4 Multiple derivatives in three dimensions

Let

$$f(x, y, z) = \frac{x^2 + y^2}{4}.$$

a) Determine $\nabla^2 f$.

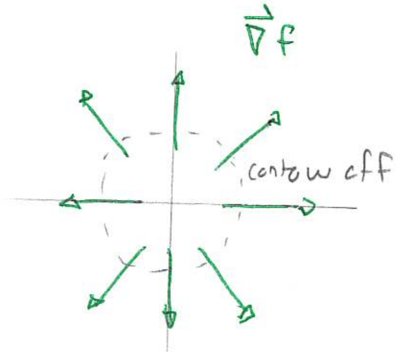
b) Determine $\nabla \times (\nabla f)$.

Answer: a) $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \cancel{\frac{\partial^2 f}{\partial z^2}}$$

$$= \frac{1}{4}\frac{\partial^2}{\partial x^2} x^2 + \frac{1}{4}\frac{\partial^2}{\partial y^2} y^2 = \frac{1}{2} + \frac{1}{2} = 1$$

b) $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}.$$

$$= \frac{x}{2}\hat{x} + \frac{y}{2}\hat{y}$$



Then

$$\vec{\nabla} \times (\vec{\nabla} f) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{2} & \frac{y}{2} & 0 \end{vmatrix} = \hat{x}\left(-\frac{\partial\, y/2}{\partial z}\right) + \hat{y}\left(\frac{\partial\, x/2}{\partial z}\right)$$

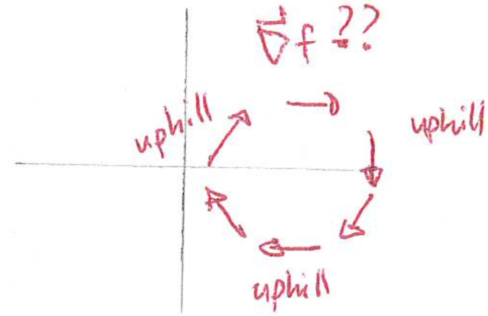
$$+ \hat{z}\left(\frac{\partial}{\partial x}\frac{y}{2} - \frac{\partial}{\partial y}\frac{x}{2}\right)$$

$$= 0$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} f) = 0$$

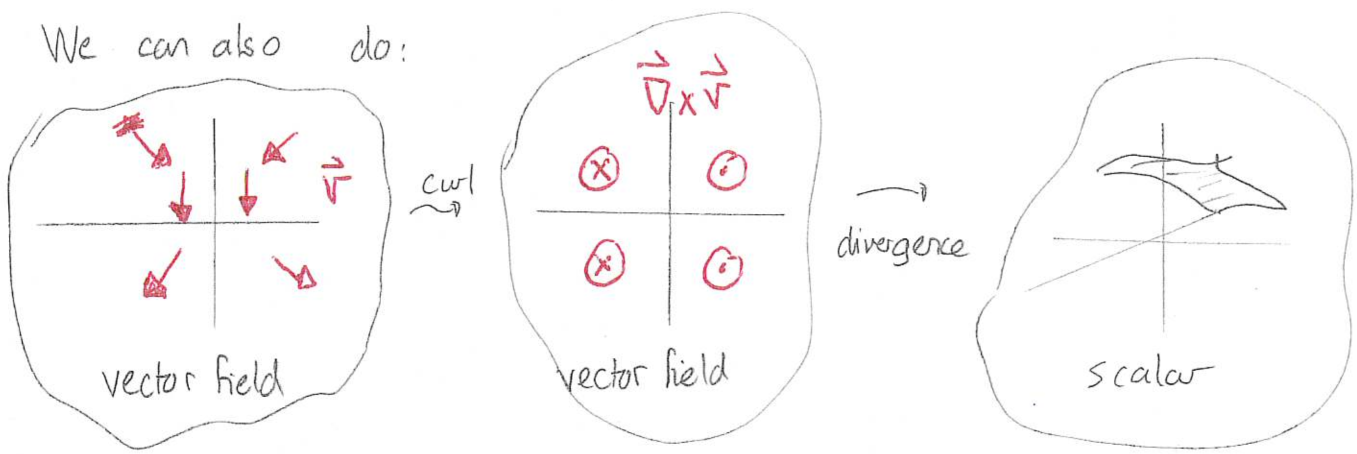
The example illustrates a general rule about gradients. A gradient cannot circle as that would indicate a closed loop along which the function keeps increasing.

One can show using standard calculus that



> For any suitably differentiable function, f,
>
> $$\vec{\nabla} \times \vec{\nabla} f = 0$$

We can also do:



Again we can show generally.

> For any suitably differentiable vector field, $\vec{v}$,
>
> $$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$$