Fri: HW 1 by 5pm

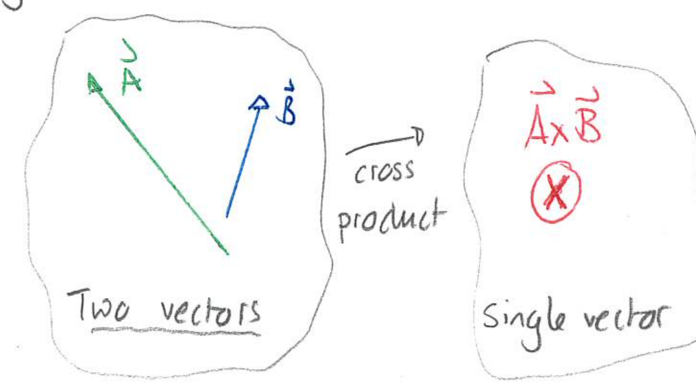
Tues: HW 1 by 5pm

Read 1.2.3, 1.2.4, 1.2.5, 1.2.6, 1.2.7

Vector Cross Product

The vector cross product is a type of multiplication that maps two vectors to another vector in an antisymmetric fashion.

This only exists for vectors in three dimensions but it appears throughout electromagnetic theory.



We can define this as:

If $\vec{A}, \vec{B}$ are two vectors, in a Cartesian basis

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

then

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

One can show that if we used a different Cartesian basis, the resulting product would ultimately give the same vector.

An alternative method to calculate the cross product is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

where the determinant is defined via

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

and

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma$$

The cross product satisfies

1) $\vec{A} \times \vec{B} = - \vec{B} \times \vec{A}$ Not commutative
2) $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
3) $(\lambda \vec{A}) \times \vec{B} = \lambda (\vec{A} \times \vec{B})$
4) $\vec{A} \times \vec{A} = 0$

Then, for orthogonal basis vectors

$\hat{x} \times \hat{x} = 0$	$\hat{x} \times \hat{y} = \hat{z}$	$\hat{y} \times \hat{x} = -\hat{z}$
$\hat{y} \times \hat{y} = 0$	$\hat{y} \times \hat{z} = \hat{x}$	$\hat{z} \times \hat{y} = -\hat{x}$
$\hat{z} \times \hat{z} = 0$	$\hat{z} \times \hat{x} = \hat{y}$	$\hat{x} \times \hat{z} = -\hat{y}$

1 Cross product algebra

The cross product satisfies

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1a}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1b}$$

$$(\lambda \mathbf{A}) \times \mathbf{B} = \lambda \mathbf{A} \times \mathbf{B} \tag{1c}$$

with similar rules when the factors are interchanged. Using only the rules of Eqs. (1) and those for the cross products of basis vectors determine $\mathbf{A} \times \mathbf{B}$ for

$$\mathbf{A} = \hat{\mathbf{x}} + 4\hat{\mathbf{y}} + 9\hat{\mathbf{z}} \quad \text{and}$$
$$\mathbf{B} = 3\hat{\mathbf{x}} + 2\hat{\mathbf{y}}.$$

Answer:

$$\vec{A} \times \vec{B} = (\hat{x} + 4\hat{y} + 9\hat{z}) \times (3\hat{x} + 2\hat{y})$$

$$= \hat{x} \times 3\hat{x} + \hat{x} \times 2\hat{y} + 4\hat{y} \times 3\hat{x} + 4\hat{y} \times 2\hat{y}$$
$$+ 9\hat{z} \times 3\hat{x} + 9\hat{z} \times 2\hat{y}$$

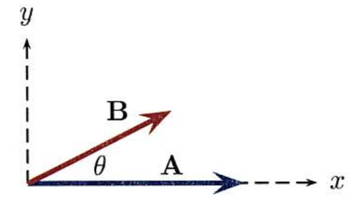
$$= 3 \underbrace{\cancel{\hat{x} \times \hat{x}}}_{0} + 2 \underbrace{\hat{x} \times \hat{y}}_{\hat{z}} + 12 \underbrace{\hat{y} \times \hat{x}}_{-\hat{z}} + 8 \overbrace{\cancel{\hat{y} \times \hat{y}}}^{0} + 27 \underbrace{\hat{z} \times \hat{x}}_{\hat{y}} + 18 \underbrace{\hat{z} \times \hat{y}}_{-\hat{x}}$$

$$= 2\hat{z} - 12\hat{z} + 27\hat{y} - 18\hat{x}$$

$$= -18\hat{x} + 27\hat{y} - 10\hat{z}$$

2 Geometry of the cross product

Two vectors in the xy plane are illustrated. Express $\mathbf{B}$ in the standard basis using B and θ and use this to determine an expression for $\mathbf{A} \times \mathbf{B}$.



$$\vec{A} = A\hat{x}$$

$$\vec{B} = B\cos\theta\,\hat{x} + B\sin\theta\,\hat{y}$$

$$\vec{A} \times \vec{B} = A\hat{x} \times \left(B\cos\theta\,\hat{x} + B\sin\theta\,\hat{y}\right)$$

$$= AB\cos\theta\,\hat{x}\times\hat{x} + AB\sin\theta\,\hat{x}\times\hat{y}$$

$$\Rightarrow \vec{A} \times \vec{B} = AB\sin\theta\,\hat{z}$$

Triple Products 1.1.3

useful! beware!

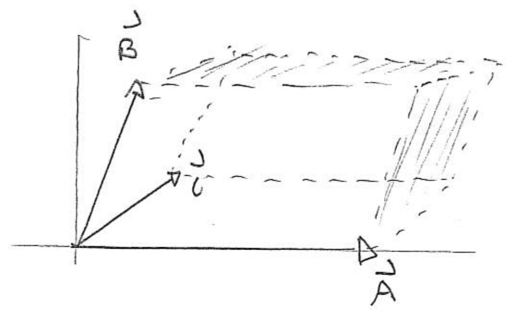
Various triple products exist and appear in electromagnetism.

1) scalar triple product

Given three vectors $\vec{A}, \vec{B}, \vec{C}$ the quantity $\vec{A} \cdot (\vec{B} \times \vec{C})$ is a scalar. One can show:

a) $$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

b) the volume of a parallel-sided box made by $\vec{A}, \vec{B}, \vec{C}$ is given by the triple product



2) vector triple product

This takes three vectors $\vec{A}, \vec{B}, \vec{C}$ and produces $\vec{A} \times (\vec{B} \times \vec{C})$. One can show

a) In general $\underbrace{\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}}_{\text{Beware}}$ although there are exceptions

b)

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

"BAC CAB" rule

Functions in three dimensions

In electromagnetism physical quantities, such as potentials and fields, depend on position. They are therefore functions of position. Examples will be:

1) charge density
2) potential

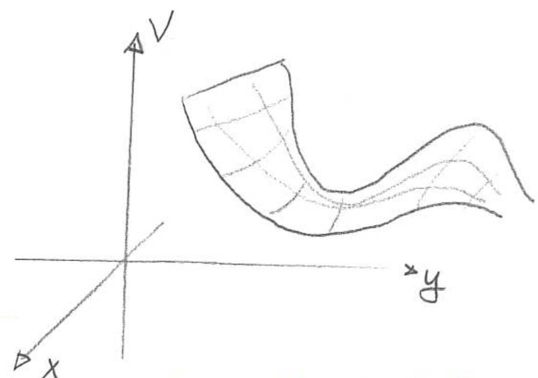
→ One number at each location

3) electric field
4) magnetic field

→ One vector at each location

We will have to consider how these vary from one location to the next and this will require calculus for such functions. The two basic types that we consider are:

1) <u>scalar functions</u> Here the function returns a single number at each location. An example is the electrostatic potential due to a point charge at the origin



two dimensional illustration

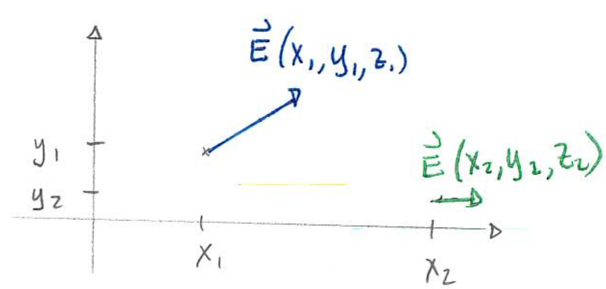
$$\underset{\text{location}}{\vec{r} \equiv (x,y,z)} \xrightarrow[V]{\text{potential}} \underbrace{V(x,y,z)}_{V(\vec{r})} = \underbrace{\frac{1}{4\pi\epsilon_0} \frac{q}{(x^2+y^2+z^2)^{1/2}}}_{\text{provide } x,y,z \text{ and this returns one scalar number.}}$$

function

q ← fixed charge

2) <u>vector functions</u>

Here the function returns one vector at each location



For example, the electric field due to charge q at the origin maps

$$\underset{\text{location}}{\vec{r} \equiv (x,y,z)} \xrightarrow{\text{function}} \vec{E} = \vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\underbrace{(x^2+y^2+z^2)^{3/2}}_{\text{number}}} \underbrace{(\overset{\text{number}}{x}\hat{x} + y\hat{y} + z\hat{z})}_{\text{vector}}$$

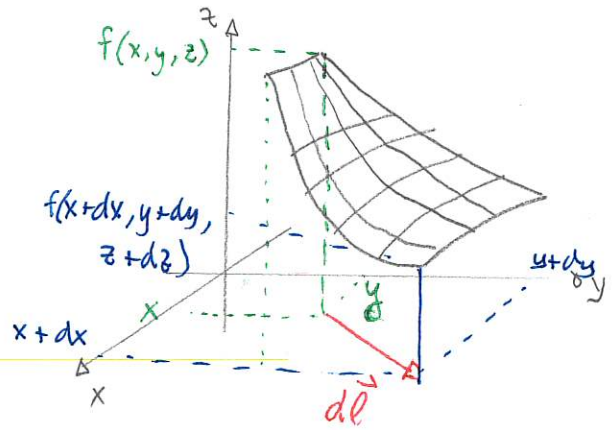
Vector functions are often called vector fields.

We will need differentiation and integration for both types of functions.

Differentiation of scalar functions

The basic idea in differentiation is to determine the rate at which a function varies as its arguments vary.

We can account for all possibilities as follows:



Choose "initial" location

(x,y,z) ← Cartesian co-ords

and evaluate

$f(x,y,z)$

Choose final location

$(x+dx, y+dy, z+dz)$

and evaluate

$f(x+dx, y+dy, z+dz$

Multivariable calculus gives

$$f(x+dx, y+dy, z+dz) \cong f(x,y,z) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

evaluate at (x,y,z)

Thus the change in f is

$$\begin{aligned} df &= f(x+dx, y+dz, z+dz) - f(x,y,z) \\ &\approx \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \end{aligned}$$

We can explore all possible changes by allowing dx, dy, dz to vary independently of each other. A compact way to express these is via the vector

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

This captures all possible nearby points to explore. We then define.

> The gradient of a function $f(x,y,z)$ is the vector
> $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

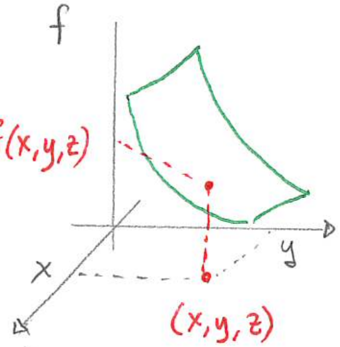
Thus, as $\vec{dl} \to 0$,

$$\boxed{df = \vec{\nabla} f \cdot \vec{dl}}$$

Only correct when co-ordinates are Cartesian

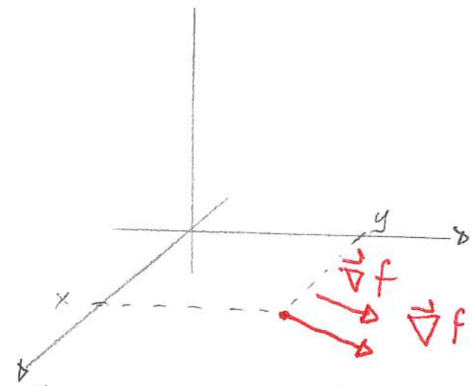
The conceptual scheme is

Given a function f we want changes df all starting at (x, y, z)



e.g. $f = \frac{1}{x^2+y^2}$

The gradient provides a vector that captures all possible changes. Calculate the vector $\vec{\nabla} f$ at each point

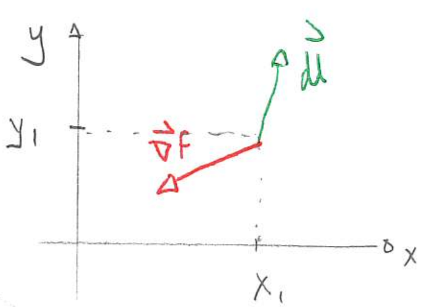


e.g. $\vec{\nabla} f = \frac{-2x}{(x^2+y^2)^2}\hat{x} - \frac{2y}{(x^2+y^2)^2}\hat{y}$

To find an actual change in f:

* describe location (x_1, y_1, z_1)
* evaluate $\vec{\nabla} f$ at (x_1, y_1, z_1)
* get direction of change of positions

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$



The change in f is

$$df \approx \vec{\nabla} f \cdot d\vec{l}$$

e.g. $d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$

$$df \approx \frac{-2x\,dx - 2y\,dy}{(x^2+y^2)^2}$$

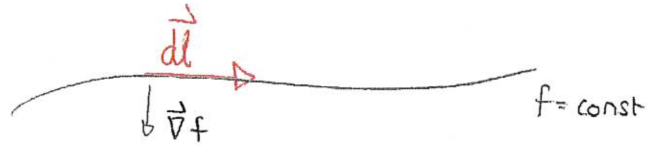
Several results can be proved

1) <u>direction of gradient</u>

The gradient of f, $\vec{\nabla} f$, is perpendicular to contours on which f is constant. It points in the direction of increasing f.

To show this consider a contour along which f is constant. Let $\vec{dl}$ be tangent to the contour.

So



$$df = 0$$

$$\Rightarrow \quad \vec{\nabla} f \cdot \vec{dl} = 0 \quad \Rightarrow \quad \vec{\nabla} f \text{ perpendicular to contour. Now}$$

$$\vec{\nabla} f \cdot \vec{dl} = df = |dl|\,|\vec{\nabla} f| \underbrace{\cos\theta}_{\theta = 0} \quad \Rightarrow \quad df > 0$$

$\rightarrow$ aligned in increasing f.

2) <u>maximum rate of change of f</u>

The maximum rate of change of f is along $\vec{\nabla} f$ and has magnitude $\lvert\vec{\nabla} f\rvert$.

To prove this:

$$\vec{\nabla} f \cdot \vec{dl} = |\vec{\nabla} f|\,|\vec{dl}| \cos\theta = df$$

Thus

$$\frac{df}{|\vec{dl}|} = |\vec{\nabla} f| \cos\theta$$

is max when $\theta = 0°$ and gives $|\vec{\nabla} f|$

3 Gradients

Let

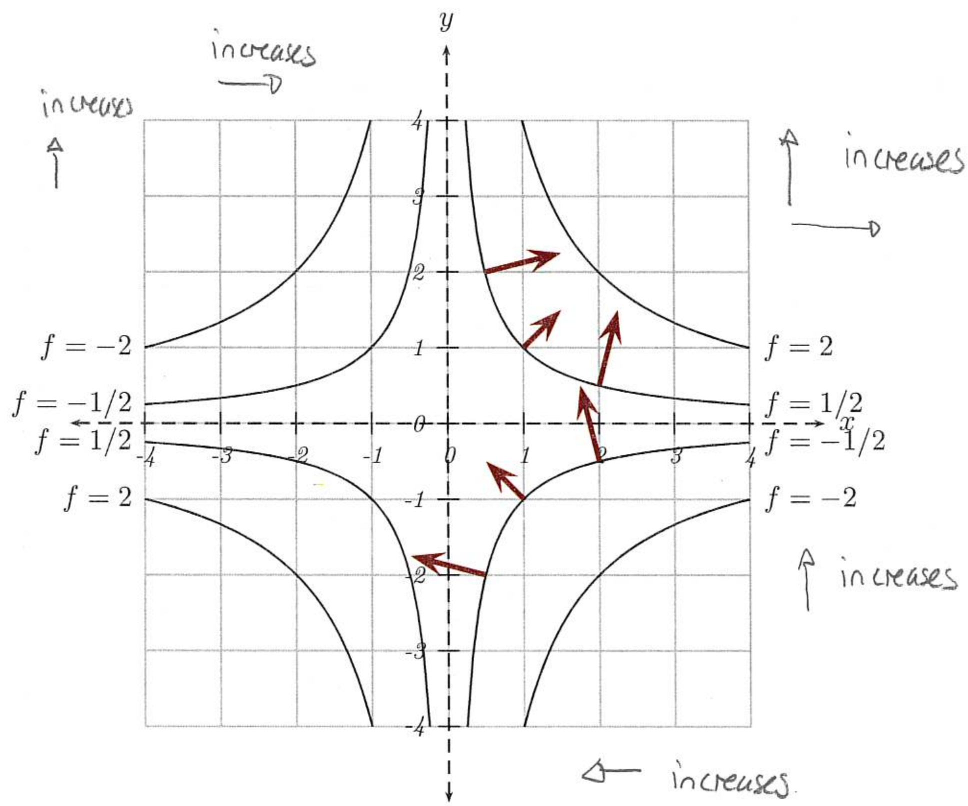
$$f(x, y) = \frac{1}{2}xy$$

be a function in two dimensions.

a) Sketch the contours of f for which $f(x, y) = 1/2$, $f(x, y) = 2$, $f(x, y) = -1/2$, and $f(x, y) = -2$. Indicate the directions in which $f(x, y)$ increases and decreases in all four quadrants.

b) Determine ∇f. Sketch the resulting vectors along the $f(x, y) = 1/2$ contour at points such as $(1, 1)$, $(2, 1/2)$, ... Repeat this for the $f(x, y) = -1/2$ contour. Are these consistent with the directions in which the function increases or decreases?

Answer:



b) $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$

$= \frac{y}{2}\hat{x} + \frac{x}{2}\hat{y}$