Electromagnetic Theory I: Homework 5

Due: 5 September 2025

This assignment will be graded immediately after the due date. If you get all problems correct, then you will receive 100%. If you have made any errors, then I will deduct 10%, point the errors out and you must submit a corrected assignment by 12 September 2025. If there are still errors, then I will deduct another 10% and you must submit the corrected assignment by 19 September 2025. This will continue until you **have solved every problem correctly.**

1 Fundamental theorem for gradients

Let

$$f(x, y, z) = x^2 y + x y^2 z.$$

a) Determine the gradient of f.

b) Determine the line integral of $\boldsymbol{\nabla} f$ over the line consisting of the straight line segments that run as follows: $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$. (See Fig. 1.28)

c) Check that the fundamental theorem for gradients for this line integral.

2 Surface integrals, drain

Consider the vector field

$$\mathbf{v} = -\mathbf{r} = -x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - z\hat{\mathbf{z}}.$$

We aim to determine $\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a}$ for a surface which consists of a cube, centered at the origin and with sides of length $2L$. In all cases let the direction of the area vector point outwards.

a) Sketch the field and use your sketch to predict whether $\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a}$ will be positive, negative or zero.

b) Determine $\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a}$ over the entire surface. Does your result match your prediction?

3 Swirling water

Suppose that water flows and this is decsribed by the velocity vector

$$\mathbf{v} = \mathbf{r} = \alpha(z\hat{\mathbf{y}} - y\hat{\mathbf{z}})$$

where α has units of s^{-1}. Consider a cubic region within the water, centered at the origin and with sides parallel to the axes and of length $2L$.

a) Without doing any calculation predict the rate at which volume of water passes through each surface of the cubic region.

b) The rate at which volume of water passes through the surface is $\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a}$ where the area vector points outward. Use this to determine the rate at which volume of water passes through each surface of the cubic region. Does the result match your prediction?