Phys 311
Fall 2019

# Electromagnetic Theory: Final Exam

11 December 2019

**Name:** Solution **Total:** **/60**

## Instructions

- There are 8 questions on 11 pages.
- Show your reasoning and calculations and always explain your answers.

## Physical constants and useful formulae

| | |
|---|---|
| Permittivity of free space | $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$ |
| Permeability of free space | $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$ |
| Charge of an electron | $e = -1.60 \times 10^{-19}\,\mathrm{C}$ |

### Integrals

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

## Question 1

A cylinder with radius $R$ extends infinitely along its axis. The charge density in the cylinder is given (in cylindrical coordinates) by

$$\rho(s', \phi', z') = \frac{3\alpha}{2\pi R^3} s'$$

where $s'$ is the distance from the cylinder axis and $\alpha$ is a constant with dimensions of C/m.

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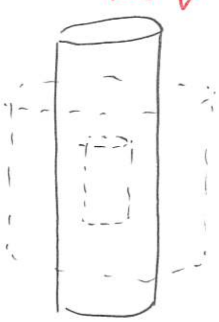
a) Determine the total charge contained in a section of the cylinder with length $L$.

$$Q = \int \rho \, d\tau = \int_0^R ds' \int_0^{2\pi} d\phi' \int_0^L dz' \, s' \rho \quad \leftarrow \frac{3\alpha}{2\pi R^3} s'$$



$$= \frac{3\alpha}{2\pi R^3} \underbrace{\int_0^R s'^2 ds'}_{R^3/3} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \underbrace{\int_0^L dz'}_{L}$$

$$\left.\begin{array}{l} 0 \le s' \le R \\ 0 \le \phi' \le 2\pi \\ 0 \le z' \le L \end{array}\right\} d\tau = s' ds' d\phi' dz' \qquad Q = \alpha L$$

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b) Determine the electric field at *any point inside* the cylinder.

In general $\vec{E} = E_s \hat{s} + E_\phi \hat{\phi} + E_z \hat{z}$

Inverting the cylinder about $x$ axis maps $E_z \to -E_z$
$E_\phi \to -E_\phi$

but distrib is same so $E_z = E_\phi = 0$. Then $E_s = E_s(s)$.

Use as a Gaussian surface a cylinder as illustrated. Then

$$\int \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

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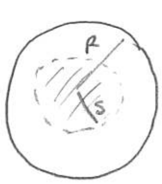


Question 1 continued ...

On the top and bottom $d\vec{a}$ is along $\hat{z}$ $\Rightarrow$ $\vec{E}\cdot d\vec{a} = 0$. So we only consider the sides. Then

$$\oint \vec{E}\cdot d\vec{a} = \int_{\text{side}} \vec{E}\cdot d\vec{a}$$

$$\left.\begin{array}{l} s' = s \\ 0 \le \phi' \le 2\pi \\ 0 \le z' \le L \end{array}\right\} \quad d\vec{a} = s\, d\phi' dz'\, \hat{s}$$

$$= \int_0^{2\pi} d\phi' \int_0^L dz'\, s\, E_s(s)$$

$$= 2\pi s L\, E_s(s).$$

So $\quad 2\pi s L E_s(s) = \dfrac{q_{enc}}{\epsilon_0}$

Now $\quad q_{enc} = \displaystyle\int_0^s ds' \int_0^{2\pi} d\phi' \int_0^L dz' s' \rho = \frac{\cancel{3}\alpha}{\cancel{2\pi} R^3} \underbrace{\frac{s^3}{\cancel{3}} \cancel{2\pi} L}_{\text{By similar calculation to a)}}$

$$= \frac{\alpha s^3}{R^3} L.$$

Thus $\quad 2\pi s \cancel{L} E_s = \dfrac{\alpha s^3}{\epsilon_0 R^3} \cancel{L}$

$$\Rightarrow \quad E_s = \frac{\alpha s^2}{2\pi \epsilon_0 R^3} \quad \Rightarrow \quad \vec{E} = \frac{\alpha s^2}{2\pi\epsilon_0 R^3}\,\hat{s}$$

c) Determine the electric field at *any point outside* the cylinder.

3 Use a Gaussian surface = cylinder radius R, height L. Then from part b)

$$2\pi s L E_s = \frac{q_{enc}}{\epsilon_0}$$

from a) $\quad q_{enc} = \alpha L$

$$\Rightarrow \quad E_s = \frac{\alpha}{2\pi\epsilon_0 s} \quad \Rightarrow \quad \vec{E} = \frac{\alpha}{2\pi\epsilon_0 s}\,\hat{s}$$

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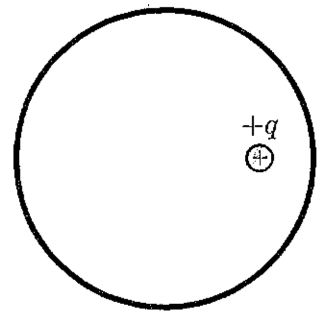
## Question 2

A spherical shell with radius $R$ contains total charge $Q > 0$ which is uniformly distributed and held fixed on the shell. A positive point charge with charge $+q$ is placed at a point *away from the center of the sphere.* Is the net force exerted by the shell on the point charge zero or not? If you say that it is not zero describe the direction of the force. Explain your answer briefly.




Inside a symmetrical spherical shell $\vec{E} = E_s \hat{s}$

Using Gauss' Law with a Gaussian sphere radius $s < R$

$$4\pi \epsilon_0 s^2 E_s = \frac{q_{enc}}{\epsilon_0} \quad (q_{enc} = 0) \Rightarrow E_s = 0$$

So $\vec{E} = 0$ inside. Then $\vec{F} = q\vec{E}$

$$\Rightarrow \boxed{\vec{F} = 0}$$

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## Question 3

Consider the following candidates for magnetic fields:

$$\mathbf{B}_1 = \beta x \hat{\mathbf{y}}$$
$$\mathbf{B}_2 = \beta x \hat{\mathbf{x}}$$

where $\beta$ is a constant. For each of these determine whether this is a possible magnetic field and, if so, the current density that could give rise to the field.

Must have $\rightarrow$ $\vec{\nabla} \cdot \vec{B} = 0$
$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Field 1

$$\vec{\nabla} \cdot \vec{B}_1 = \cancel{\frac{\partial B_{1x}}{\partial x}}^{\,0} + \frac{\partial B_{1y}}{\partial y} + \cancel{\frac{\partial B_{1z}}{\partial z}}^{\,0}$$

$$= \frac{\partial}{\partial y} \beta x = 0$$

This is a possible field.

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \beta x & 0 \end{vmatrix} = \beta \hat{z}$$

$$\Rightarrow \vec{J} = \frac{\beta}{\mu_0} \hat{z}$$

Field 2

$$\vec{\nabla} \cdot \vec{B}_2 = \frac{\partial}{\partial x}(\beta x) = \beta$$

Unless $\beta = 0$ this is not a field

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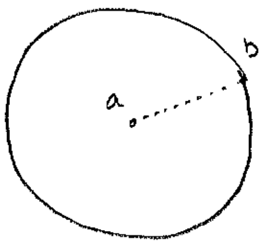
## Question 4

A solid sphere with radius $R$ contains charge $Q > 0$ that is uniformly distributed. The resulting electric field (given in spherical coordinates) is

$$\mathbf{E} = \begin{cases} \dfrac{Qr}{4\pi\epsilon_0 R^3}\,\hat{\mathbf{r}} & \text{if } 0 \leqslant r \leqslant R \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}} & \text{if } r \geqslant R. \end{cases}$$

Determine the electrostatic potential difference between the center of the sphere and the edge of the sphere, indictating which is at the higher potential.




$$\Delta V = V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{l}$$

Take as a path a line radially out from the center. So

$$d\vec{l} = dr\,\hat{r} \qquad 0 \leq r \leq R$$

Then $\vec{E}\cdot d\vec{l} = \dfrac{Qr}{4\pi\epsilon_0 R^3}\,dr$ and

$$\Delta V = -\frac{Q}{4\pi\epsilon_0 R^3}\underbrace{\int_0^R r\,dr}_{R^2/2} = -\frac{Q}{8\pi\epsilon_0 R}$$

So $V_{edge} - V_{center} = -\dfrac{Q}{8\pi\epsilon_0 R}$ negative $\Rightarrow$ $V_{center}$ is higher.

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## Question 5

In the following question, **either do part a) or part b) for full credit.**

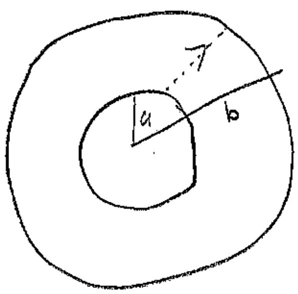
a) Two spherical conducting shells are centered at the same point. The inner shell has radius $a$ and total charge $Q$. The outer shell has radius $b > a$ and total charge $-Q$. The electric field *between the two shells* is, in spherical coordinates,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}$$

Determine the capacitance of the pair of shells.

Need $Q = C\Delta V$ and we need $\Delta V$ via

$$\Delta V = -\int \vec{E}\cdot d\vec{l}$$




use radial path $a \le r \le b$

$$d\vec{l} = dr\,\hat{r}$$

So $\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,dr$

$$\Delta V = -\frac{1}{4\pi\epsilon_0}\int_a^b \frac{Q}{r^2}\,dr$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)$$

so $Q = \dfrac{4\pi\epsilon_0}{1/b - 1/a}\,\Delta V$

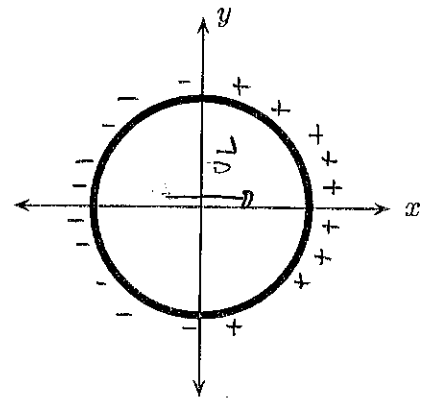
$$\Rightarrow \quad C = \frac{4\pi\epsilon_0}{1/b - 1/a}$$

Question 5 continued ...

b) A one-dimensional circular loop with radius $R$ carries charge with linear charge density (given in cylindrical coordinates)

$$\lambda(s', \phi', z') = \lambda \cos(\phi').$$

Determine the electric dipole moment for this distribution.




In general

$$\vec{p} = \int \rho(\vec{r}\,')\, \vec{r}\,' d\tau'$$

For one dimension

$$\vec{p} = \int \lambda(\vec{r}\,')\, \vec{r}\,' dl'$$

$r' = R$

Here $dl' = R\, d\phi'$ where $0 \leq \phi' \leq 2\pi$

$$\vec{r}\,' = r' \hat{s}$$
$$= R(\cos\phi' \hat{x} + \sin\phi' \hat{y})$$

So
$$\vec{p} = \lambda R \int_0^{2\pi} \cos\phi' (\cos\phi' \hat{x} + \sin\phi' \hat{y})\, d\phi'$$

$$= \lambda R \left\{ \underbrace{\int_0^{2\pi} \cos^2\phi' d\phi'}_{\pi} \hat{x} + \underbrace{\int_0^{2\pi} \cos\phi' \sin\phi' d\phi'}_{0} \hat{y} \right\}$$

$$\Rightarrow \vec{p} = \lambda R \pi \hat{x}$$

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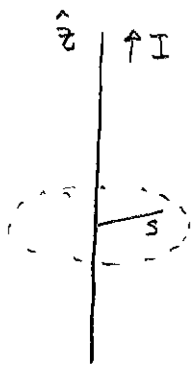
## Question 6

An infinitely long straight wire carries uniform and constant current $I$. Starting from an appropriate first principle (e.g. Biot-Savart law, Ampère's law, or magnetic vector potential), determine the magnetic field produced by the current at all points.




In general

$$\vec{B} = B_s \hat{s} + B_\phi \hat{\phi} + B_z \hat{z}$$

By Biot-Savart law $\vec{B}$ cannot have a component along $\hat{z}$ $\Rightarrow B_z = 0$

Inverting about x axis maps $\vec{B} \to B_s \hat{s} - B_\phi \hat{\phi}$ and flips current. Should get $\vec{B} = -B_s \hat{s} - B_\phi \hat{\phi}$

$\Rightarrow B_s = 0$

Thus $\vec{B} = B_\phi(s)\,\hat{\phi}$

Use as an Amperian loop a circle radius $s$

Then $0 \leq \phi' \leq 2\pi$ $\quad \vec{dl} = s\,d\phi'\,\hat{\phi}$.

and

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

$$\Rightarrow \int_0^{2\pi} s\,B_\phi(s)\,d\phi' = \mu_0 I$$

$$s B_\phi(s) \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} = \mu_0 I \quad \Rightarrow \quad B_\phi(s) = \frac{\mu_0 I}{2\pi s}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{s}$$

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## Question 7

In the following question, **either do part a) or part b) for full credit.**

a) An electron is placed between two parallel plates that carrying uniform charge densities and each of which move with the same constant velocities $\mathbf{v} = v_{\text{plate}}\hat{\mathbf{x}}$. The configuration of these results in an electric field between the plates given by

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}}$$

and a magnetic field between them given by

$$\mathbf{B} = \mu_0 \sigma v_{\text{plate}}\,\hat{\mathbf{y}}$$

where $\sigma$ is the charge per unit area on each plate. Suppose that an electron is between the plates, and at one instant, moves with velocity $\mathbf{v}_{\text{elec}} = v_{\text{elec}}\hat{\mathbf{x}}$. Determine an expression for the net force exerted on the electron. Is it possible for the magnetic force to be larger than the electric force (this requires the result that $\epsilon_0\mu_0 = 1/c^2$ where $c$ is the speed of light)? Explain your answer.

$$\vec{F} = \vec{F}_{elec} + \vec{F}_{mag}$$

$$\vec{F}_{elec} = q_{elec}\vec{E} = -e\frac{\sigma}{\epsilon_0}\hat{z} \implies \vec{F}_{elec} = -e\frac{\sigma}{\epsilon_0}\hat{z}$$

$$\vec{F}_{mag} = q(\vec{v}\times\vec{B})$$

$$= -e\left[V_{elec}\hat{x} \times \mu_0\sigma V_{plate}\hat{y}\right]$$

$$= -e V_{elec}\mu_0\sigma V_{plate}\hat{z} \implies \vec{F}_{mag} = -e\mu_0 V_{elec}V_{plate}\sigma\hat{z}$$

$$\vec{F} = -e\sigma\left[\frac{1}{\epsilon_0} + V_{elec}V_{plate}\mu_0\right]\hat{z}$$

For $F_{mag} > F_{elec}$ need $\cancel{e}\mu_0 V_{elec}V_{plate}\cancel{\sigma} > \cancel{e}\cancel{\sigma}/\epsilon_0$

$$\implies V_{elec}V_{plate} > \frac{1}{\mu_0\epsilon_0} = c^2$$

Question 7 continued ...

This requires

$V_{elec}$ or $V_{plate} > c$

impossible.

b) Consider two concentric conducting spheres with radii $a$ and $b$ where $a < b$. The region between these spheres is filled with an Ohmic material with conductivity $\sigma$. The outer sphere is held a higher potential than the inner sphere and the electric potential in the region between these is

$$V(r) = b\frac{\Delta V}{b-a}\left(1 - \frac{a}{r}\right) + V(a)$$

where $\Delta V$ is the potential difference between the spheres. Determine an expression for the current density $\mathbf{J}$ in the region between the spheres and use this to determine the current that flows from one sphere to the other. The answers should be in terms of $a, b, \sigma, \Delta V$ and constants.

In general $\vec{J} = \sigma \vec{E}$ and we need $\vec{E} = -\vec{\nabla} V$

$$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r} - \cancel{\frac{1}{r}\frac{\partial V}{\partial \theta}}\hat{\theta} - \cancel{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}}\hat{\phi}$$

$$= -b\frac{\Delta V}{b-a}\frac{\partial}{\partial r}\left(1 - \frac{a}{r}\right)\hat{r}$$

$$= -b\frac{\Delta V}{b-a}\left(\frac{a}{r^2}\right)\hat{r}$$

$$\vec{E} = -ab\frac{\Delta V}{b-a}\frac{1}{r^2}\hat{r} \Rightarrow \vec{J} = -ab\sigma\frac{\Delta V}{b-a}\frac{1}{r^2}\hat{r}$$

To get current determine current flowing through a sphere with radius $a < r \leq b$, Then

$$I = \int \vec{J}\cdot d\vec{a} \qquad d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$= -\frac{ab\sigma\Delta V}{b-a}\int_0^{\pi}\sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$I = -\frac{ab\sigma 4\pi \Delta V}{b-a}$$ flows inward

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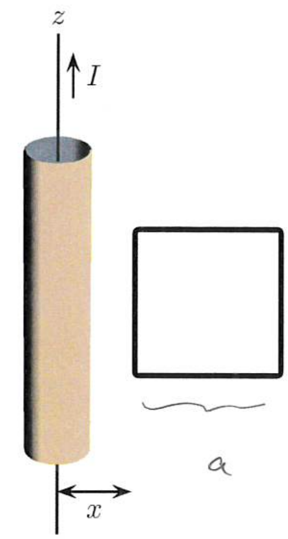
## Question 8

An infinitely long cylinder lies along the $z$ axis and this carries a uniformly distributed current $I$ along its length. The magnetic field produced by this *outside* the cylinder is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

where $s$ is the distance from the axis of the cylinder. A square loop is placed with its vertical edge aligned in the same direction as the cylinder axis and its horizontal edge pointing straight out. The length of the side of the loop is $a$. The left edge of the loop is initially distance $x$ from the cylinder axis.




a) Determine an expression for the flux through the loop.

$\Phi = \int \vec{B}\cdot d\vec{a}'$ For the loop $\left.\begin{array}{l} x \le s' \le x+a \\ 0 \le z' \le a \end{array}\right\}$ $d\vec{a} = ds'dz'\hat{\phi}$

So $\vec{B}\cdot d\vec{a}' = \frac{\mu_0 I}{2\pi s'}\, ds'dz'$

$$\Phi = \int_x^{x+a} ds' \int_0^a dz' \frac{\mu_0 I}{2\pi s'} = \frac{\mu_0 I}{2\pi} \underbrace{\int_x^{x+a} \frac{ds'}{s'}}_{\ln\frac{x+a}{x}} \underbrace{\int_0^a dz'}_{a}$$

$$\Phi = \frac{\mu_0 I a}{2\pi} \ln\left(\frac{x+a}{x}\right)$$

b) Suppose that the loop is dragged directly away from the cylinder with speed $v$. Determine an expression for the EMF around the loop at the illustrated instant.

$v = \frac{dx}{dt}$ $\qquad \mathcal{E} = -\frac{d\Phi}{dt}$

$$= -\frac{\mu_0 I a}{2\pi}\frac{d}{dt}\ln\left(\frac{x+a}{x}\right) = -\frac{\mu_0 I a}{2\pi}\frac{d}{dt}\left[\ln(x+a) - \ln x\right]$$

$$= -\frac{\mu_0 I a}{2\pi}\left[\frac{1}{x+a} - \frac{1}{x}\right]\underbrace{\frac{dx}{dt}}_{v}$$

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$\Rightarrow \mathcal{E} = \frac{\mu_0 I a}{2\pi}\frac{-a}{x(x+a)}\, v$