Final Exam: Tuesday, May 16, 8 $\to$ 9:50am.

* Covers: Entire Semester
* Bring: A total of one letter sheet - single side
* Given info: See previous final.
* Review: 2007 All questions
  2022 All questions

General state for two spin-½ particles

We saw that for two spin-½ particles, there exist special states of the form

$$|+\hat{z}\rangle_A |+\hat{z}\rangle_B$$
$$|+\hat{z}\rangle_A |-\hat{z}\rangle_B$$
$$|-\hat{z}\rangle_A |+\hat{z}\rangle_B$$
$$|-\hat{z}\rangle_A |-\hat{z}\rangle_B$$

and these can be extended to product states such as

$$|\Psi\rangle = |+\hat{x}\rangle_A |+\hat{y}\rangle_B \quad \Rightarrow \quad |\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+\hat{z}\rangle + |-\hat{z}\rangle\right)\frac{1}{\sqrt{2}}\left(|+\hat{z}\rangle + i|-\hat{z}\rangle\right)$$

$$= \frac{1}{2}\left[|+\hat{z}\rangle|+\hat{z}\rangle + i|+\hat{z}\rangle|-\hat{z}\rangle + |-\hat{z}\rangle|+\hat{z}\rangle + i|-\hat{z}\rangle|-\hat{z}\rangle\right]$$

Measure S_x on A $\Rightarrow +\hbar/2$ with certainty

Measure S_y on B $\Rightarrow +\hbar/2$ with certainty

The most general product state has the form

$$|\Psi\rangle = \underbrace{|\psi_1\rangle_A}_{\text{state for A}} \underbrace{|\psi_2\rangle_A}_{\text{state for B}} \quad \Rightarrow \quad |\Psi\rangle = |+\hat{m}\rangle|+\hat{n}\rangle$$

for some unit vectors $\hat{m}, \hat{n}$

Then quantum theory dictates that any products are possible and the general state is

$$\boxed{|\Psi\rangle = a_0|+z\rangle|+z\rangle + a_1|+z\rangle|-z\rangle + a_2|-z\rangle|+z\rangle + a_3|-z\rangle|-z\rangle}$$

where

$$|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 = 1.$$

We can ask

> Given any state $|\Psi\rangle$ is it possible to express this as a product of states for each particle?

$$\Downarrow$$

> There exists a measurement on A alone and a measurement on B alone such that each gives one particular outcome with certainty.

If this is possible then

$$|\Psi\rangle = \left(b_0|+z\rangle + b_1|-z\rangle\right)\left(c_0|+z\rangle + c_1|-z\rangle\right)$$

for some b_0, b_1, c_0, c_1 such that

$$|b_0|^2 + |b_1|^2 = 1$$

$$|c_0|^2 + |c_1|^2 = 1$$

1 Singlet state

Consider the following possible state for two spin-1/2 particles.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|+\hat{z}\rangle|-\hat{z}\rangle - |-\hat{z}\rangle|+\hat{z}\rangle\right].$$

a) Determine whether this can be written as a product state or not.

b) Show that for any unit vector $\hat{\mathbf{n}}$ this state can be expressed as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|+\hat{n}\rangle|-\hat{n}\rangle - |-\hat{n}\rangle|+\hat{n}\rangle\right],$$

up to an overall phase.

c) Suppose that S_z is measured for each particle. List the possible pairs of outcomes and the probabilities with which they appear.

d) Suppose that S_x is measured for each particle. List the possible pairs of outcomes and the probabilities with which they appear.

e) How do the previous results extend to any component of spin? What does this suggest for the component of the total spin of the system?

f) Can you see a pattern with the outcomes if the same component is measured for each particle?

Answer: a) Assume that the state is a product

$$|\Psi\rangle = \left(b_0|+z\rangle + b_1|-z\rangle\right)\left(c_0|+z\rangle + c_1|-z\rangle\right)$$

$$= b_0c_0|+z\rangle|+z\rangle + b_0c_1|+z\rangle|-z\rangle + b_1c_0|-z\rangle|+z\rangle + b_1c_1|-z\rangle|-z\rangle$$

So we need $b_0c_0 = 0$ $\quad b_0c_1 = \frac{1}{\sqrt{2}}$

$b_1c_1 = 0$ $\quad b_1c_0 = \frac{1}{\sqrt{2}}$

Suppose $b_0 \neq 0$ then $c_0 = 0$ but this contradicts $b_1c_0 = \frac{1}{\sqrt{2}}$. Thus $b_0 = 0$. This contradicts $b_0c_1 = \frac{1}{\sqrt{2}}$.

Thus the state is not a product.

b) $|+\hat{n}\rangle = \cos\frac{\theta}{2}|+\hat{z}\rangle + e^{i\phi}\sin\theta/2\,|-\hat{z}\rangle$

$|-\hat{n}\rangle = \sin\theta/2\,|+z\rangle - e^{i\phi}\cos\theta/2\,|-z\rangle$

Thus

$$|+\hat{n}\rangle|-\hat{n}\rangle = \cos\theta/2\,\sin\theta/2\,|+z\rangle|+z\rangle - e^{i\phi}\cos^2\theta/2\,|+z\rangle|-z\rangle$$
$$+ e^{i\phi}\sin^2\theta/2\,|-z\rangle|+z\rangle - e^{2i\phi}\sin\theta/2\cos\theta/2\,|-z\rangle|-z\rangle.$$

$$|-\hat{n}\rangle|+\hat{n}\rangle = \sin\theta/2\,\cos\theta/2\,|+z\rangle|+z\rangle + e^{i\phi}\sin^2\theta/2\,|+z\rangle|-z\rangle$$
$$- e^{i\phi}\cos^2\theta/2\,|-z\rangle|+z\rangle - e^{2i\phi}\sin\theta/2\cos\theta/2\,|-z\rangle|-z\rangle.$$

Thus

$$|+\hat{n}\rangle|-\hat{n}\rangle - |-\hat{n}\rangle|+\hat{n}\rangle = -2e^{i\phi}\underbrace{(\cos^2\theta/2 + \sin^2\theta/2)}_{1}|+z\rangle|-z\rangle$$
$$+2e^{i\phi}\underbrace{(\cos^2\theta/2 + \sin^2\theta/2)}_{1}|-z\rangle|+z\rangle$$

$$= -2e^{i\phi}\left[|+z\rangle|-z\rangle - |-z\rangle|+z\rangle\right]$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left[|+n\rangle|-n\rangle - |-n\rangle|+n\rangle\right] = -e^{i\phi}\frac{1}{\sqrt{2}}\left[|+z\rangle|-z\rangle - |-z\rangle|+z\rangle\right]$$

$$= e^{i(\phi+\pi)}\frac{1}{\sqrt{2}}\left[|+z\rangle|-z\rangle - |-z\rangle|+z\rangle\right]$$

c) We calculate probabilities via $|\langle +z|\langle +z|\Psi\rangle|^2$ etc.... Then

	Particle A	Particle B	Probabilities
$S_z =$	$+\hbar/2$	$+\hbar/2 = S_z$	0
	$+\hbar/2$	$-\hbar/2$	$1/2$
	$-\hbar/2$	$+\hbar/2$	$1/2$
	$-\hbar/2$	$-\hbar/2$	0

d) For this:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|+\hat{x}\rangle|-\hat{x}\rangle - |-\hat{x}\rangle|+\hat{x}\rangle\right]$$

and then

Particle A	Particle B	Probabilities
S_x	S_x	
$+\hbar/2$	$+\hbar/2$	0
$+\hbar/2$	$-\hbar/2$	$1/2$
$-\hbar/2$	$+\hbar/2$	$1/2$
$-\hbar/2$	$-\hbar/2$	0

e) Suppose we measure the $\hat{n}$ component for each particle. We will always get exactly the opposite outcome for the two particles

S_n for A	S_n for B	Prob
$+\hbar/2$	$+\hbar/2$	0
$+\hbar/2$	$-\hbar/2$	$1/2$
$-\hbar/2$	$+\hbar/2$	$1/2$
$-\hbar/2$	$-\hbar/2$.	0

The n component of spin $S_n(A) + S_n(B) = 0$ always.

f) The outomes are always exacly opposite.

Entangled states

Any state $|\Psi\rangle$ that canot be expressed as a product state is called an entangled state. Such entangled states have the properties:

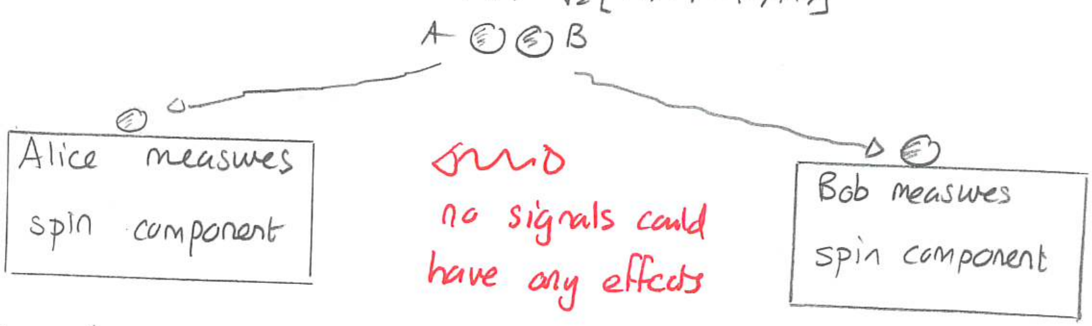
1) there is no pair of measurements, one on each particle which will yield one pair of outcomes with certainty.

2) measurement outcomes in general are correlated.

EPR paradox

The singlet state can be used to construct a paradox originally due to Einstein, Podosky and Rosen. This concerns particles prepared in a singlet state. Each particle is transmitted to an observer who can measure any spin-component

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle|-\rangle - |-\rangle|+\rangle \right]$$



When these measurements are done, the particles are separated by a distance that prohibits any signalling or communication

2 EPR paradox

Consider two spin-1/2 particles in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|+\hat{z}\rangle\,|-\hat{z}\rangle - |-\hat{z}\rangle\,|+\hat{z}\rangle\right].$$

Suppose that observer A measures the left particle and B measures the right particle.

a) If A were to measure S_z, could A predict the outcome?

b) If B were to measure S_z, could B predict the outcome?

c) Suppose that A measured S_z and obtained $S_z = +\hbar/2$. If B were to subsequently measure S_z could A (or B or anyone else predict the outcome)? Does A's action appear to have affected B's particle?

d) Suppose that A measured S_z and obtained $S_z = -\hbar/2$. If B were to subsequently measure S_z could A (or B or anyone else predict the outcome)? Does A's action appear to have affected B's particle?

e) Suppose that A measured S_x and obtained $S_x = +\hbar/2$. If B were to subsequently measure S_x could A (or B or anyone else predict the outcome)? Does A's action appear to have affected B's particle? How does this extend to any component?

f) Does it appear that there is a definite value for S_z for B prior to measurement? What about S_x?

Answer: a) The probability table is

$S_z(A)$	$S_z(B)$	Prob
$+\hbar/2$	$+\hbar/2$	0
$+\hbar/2$	$-\hbar/2$	1/2
$-\hbar/2$	$+\hbar/2$	1/2
$-\hbar/2$	$-\hbar/2$	0

$\Rightarrow$ A could get either outcome with equal probability

$\Rightarrow$ A could not predict.

b) Similarly B could not predict.

c) In this case B would definitely get $-\hbar/2$. Anyone can predict

d) " " " " " " get $+\hbar/2$ " " "

In both cases A's action seems to have affected B's particle.

e) The situation is the same as for the z components. And any other.

f) It does since if A gets a particular outcome that fixes B's outcome. But A could not have communicated with B. Same for S_x

The EPR paradox then asserts that because A could not have communicated with B then B must have :

1) a predetermined (but unknown) value for S_z
2) " " (" ") " " S_x.

But quantum theory gives $[\hat{S}_x, \hat{S}_z] \neq 0$ and so this is not generally possible.

The entangled singlet pair state appears to contradict this aspect of quantum theory.

Bell inequalities

The possible explanations of such situations can be assessed via correlations. For each observer we could assign values to the outcomes

$$e^- \rightarrow \boxed{\text{measure } S_n} \begin{cases} \rightarrow S_n = +\hbar/2 & \rightsquigarrow \text{assign outcome } +1 \\ \rightarrow S_n = -\hbar/2 & \rightsquigarrow \text{assign outcome } -1 \end{cases}$$

We then do the following

$e^- \rightarrow$ A measures S_a for direction $\hat{a}$ $\rightarrow$ A assigns value $A(\hat{a})$

$e^- \rightarrow$ B measures S_b for direction $\hat{b}$ $\rightarrow$ B assigns value $B(\hat{b})$

$\rightarrow$ Form the product $A(\hat{a})\, B(\hat{b})$

$\rightarrow$ Average over many runs $\langle A(\hat{a})\, B(\hat{b}) \rangle = E(\vec{a}, \vec{b})$

3 Bell Correlations

Consider two spin-1/2 particles in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|+\hat{z}\rangle\,|-\hat{z}\rangle - |-\hat{z}\rangle\,|+\hat{z}\rangle\right].$$

Suppose that observer A measures the left particle and B measures the right particle.

a) Suppose that each measures S_z. What does quantum theory predict for $E(\hat{\mathbf{z}},\hat{\mathbf{z}}) = \langle A(\hat{\mathbf{z}})B(\hat{\mathbf{z}})\rangle$?

b) Suppose that each measures S_x. What does quantum theory predict for $E(\hat{\mathbf{x}},\hat{\mathbf{x}}) = \langle A(\hat{\mathbf{x}})B(\hat{\mathbf{x}})\rangle$?

c) Suppose that A measures S_z and B measures S_x. What does quantum theory predict for $E(\hat{\mathbf{z}},\hat{\mathbf{x}}) = \langle A(\hat{\mathbf{z}})B(\hat{\mathbf{x}})\rangle$?

d) Suppose that A measures S_z and B measures S_n where $\hat{\mathbf{n}} = \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}$. What does quantum theory predict for $E(\hat{\mathbf{z}},\hat{\mathbf{n}}) = \langle A(\hat{\mathbf{z}})B(\hat{\mathbf{n}})\rangle$?

Answer: a)

$A(\hat{z})$	$B(\hat{z})$	$A(\hat{z})B(\hat{z})$	Prob
+1	+1	+1	0
+1	−1	−1	1/2
−1	+1	−1	1/2
−1	−1	+1	0

$E(\hat{z},\hat{z}) = \frac{1}{2}(-1) + \frac{1}{2}(-1) = -1$

b) The table will be identical to a) $E(\hat{x},\hat{x}) = -1.$

c)

$A(\hat{z})$	$B(\hat{x})$	$A(\hat{z})B(\hat{x})$	Prob
+1	+1	+1	1/4
+1	−1	−1	1/4
−1	+1	−1	1/4
−1	−1	+1	1/4

$E(\hat{x},\hat{z}) = \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = 0$

d) We need $\text{Prob}(S_z = +\hbar/2, S_n = +\hbar/2) = |\langle +z|\langle +\hat{n}|\Psi\rangle|^2$

$$= \left|\frac{1}{\sqrt{2}}\langle +\hat{n}|-\hat{z}\rangle\right| = \frac{1}{2}\left|e^{i\phi}\sin\theta/2\right|^2$$

$$= \frac{1}{2}\sin^2\theta/2$$

$$\text{Prob}(S_z = +\hbar/2, S_n = -\hbar/2) = \left|\frac{1}{\sqrt{2}}\langle -\hat{n}|-\hat{z}\rangle\right|^2 = \frac{1}{2}\cos^2\theta/2$$

The table becomes

$A(z)$	$B(n)$	$A(z)\,B(n)$	Prob
+1	+1	+1	$\frac{1}{2}\sin^2\theta/2$
+1	−1	−1	$\frac{1}{2}\cos^2\theta/2$
−1	+1	−1	$\frac{1}{2}\cos^2\theta/2$
−1	−1	+1	$\frac{1}{2}\sin^2\theta/2$

$$E(\hat{z},\hat{n}) = -\cos^2\theta/2 + \sin^2\theta/2 = -\cos\theta$$

In general quantum theory predicts that for any directions $\hat{a}, \hat{b}$

$$\boxed{E(\hat{a},\hat{b}) = -\,\hat{a}\cdot\hat{b}}$$

Bell's inequalities considers a situation where

Observer A randomly chooses directions $\hat{a}, \hat{a}'$

Observer B randomly " " $\hat{b}, \hat{b}'$

They measure and form

$$\Lambda = E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') + E(\hat{a}',\hat{b}) + E(\hat{a}',\hat{b}')$$

Then a local realistic variable model assumes that in any experimental run the quantities $A(\hat{a}), A(\hat{a}')$ $B(\hat{b}), B(\hat{b}')$ are predetermined. So we construct Λ via

$$A(\hat{a})\,B(\hat{b}) - A(\hat{a})\,B(\hat{b}') + A(\hat{a}')\,B(\hat{b}) + A(\hat{a}')\,B(\hat{b}')$$

$$= A(\hat{a})\left[B(\hat{b}) - B(\hat{b}')\right] + A(\hat{a}')\left[B(\hat{b}) - B(\hat{b}')\right]$$

one of these is ± 2 the other zero

these are ± 1

This quantity is banded by ± 2. Thus we expect

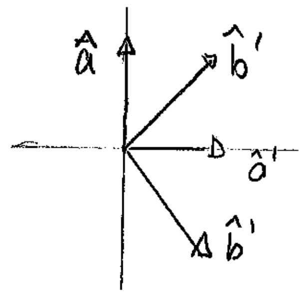
$$\left| E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') + E(\hat{a}',\hat{b}) + E(\hat{a}',\hat{b}') \right| \leq 2$$

Quantum theory predicts that, for the singlet state

$$\left| E(\hat{a},\hat{b}) - \dots \dots \right|$$

$$= \left| \hat{a}\cdot\hat{b} - \hat{a}\cdot\hat{b}' + \hat{a}'\cdot\hat{b} + \hat{a}'\cdot\hat{b}' \right|$$

Then choosing the measurement directions



gives:

$$\left| E(\hat{a},\hat{b}) - \dots \right|$$

$$= \left| \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right| = 4/\sqrt{2}$$

$$= 2\sqrt{2}$$

Thus

local realistic $\Rightarrow$ correlation combination ≤ 2
quantum theory $\Rightarrow$ " " > 2

We can, and these have been, checked by experiments