Thurs: Read 7.4, 7.5, 7.6

Fri: HW 5pm

Angular momentum eigenvalues and eigenstates

The angular momentum operators are:

$\hat{J}_x \rightsquigarrow$ x - component angular momentum

$\hat{J}_y \rightsquigarrow$ y - " " "

$\hat{J}_z \rightsquigarrow$ z - " " "

$\hat{\vec{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \rightsquigarrow$ magnitude squared angular momentum

These satisfy:

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$$

$$[\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x$$

$$[\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y$$

$$[\hat{\vec{J}}^2, \hat{J}_i] = 0$$

$\rightarrow$ There exist simultaneous eigenstates of $\hat{\vec{J}}^2$ and $\hat{J}_z$ and these are denoted

$$|j,m\rangle$$

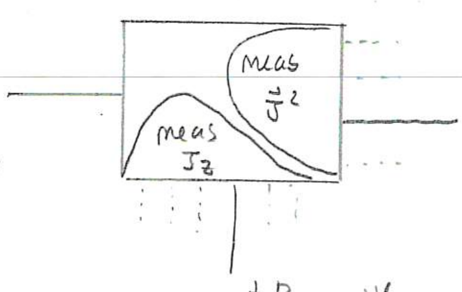
with eigenvalue equations

$$\hat{\vec{J}}^2 \, |j,m\rangle = \hbar^2 a_j(a_j+1)\, |j,m\rangle$$

$$\hat{J}_z \, |j,m\rangle = \hbar B_m |j,m\rangle$$

Physically

$|j,m\rangle$



$\hbar^2 a_j(a_j+1)$ with certainty

$\hbar B_m$ with certainty

The algebra of the angular momentum operators gives:

1) The magnitude of $\vec{J}^2$ is positive:

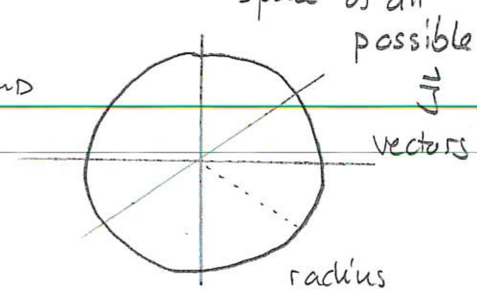
$$a_j > 0$$

2) the range of J_z is restricted, depending on a_j. Specifically

$$-a_j \leq B_m \leq a_j$$

Physically we can view this in a semi-classical way as:

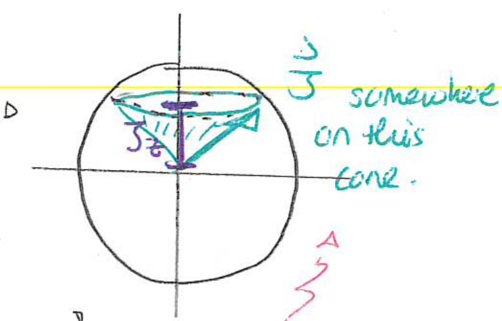
The index j determines via a_j the magnitude $\vec{J}^2$. This would correspond to a $\vec{J}$ vector somewhere on a sphere with radius $\hbar\sqrt{a_j(a_j+1)}$



The index m determines, via B_m, the z-component of $\vec{J}$, i.e.

$$J_z = \hbar B_m$$

It is always true that $J_z < |\vec{J}|$ and $(\Delta J_x)^2 + (\Delta J_y)^2 > 0$



It is impossible that $J_z = \pm|\vec{J}|$ or $J_x = \pm|\vec{J}|$ ...

the most precise we could be about the vector $\vec{J}$.

Angular momentum raising and lowering operators

The angular momentum raising and lowering operators are defined as:

$$\boxed{\begin{aligned} \hat{J}_+ &= \hat{J}_x + i\hat{J}_y \\ \hat{J}_- &= \hat{J}_x - i\hat{J}_y \end{aligned}}$$

$\leftarrow$ raising operator

$\leftarrow$ lowering operator.

Then we can use operator algebra to show:

1) $\hat{J}_+^\dagger = \hat{J}_-$ and $\hat{J}_-^\dagger = \hat{J}_+$

2) $[\hat{J}_z, \hat{J}_\pm] = \pm\hbar \hat{J}_\pm$

$[\hat{J}^2, \hat{J}_\pm] = 0$

3) $[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$

4) $\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar \hat{J}_z$

$\hat{J}^2 = \hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar \hat{J}_z$

1 Angular momentum ladder operators

The angular momentum raising and lowering operators are:

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y$$
$$\hat{J}_- = \hat{J}_x - i\hat{J}_y$$

a) Show that

$$[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm.$$

b) Show that

$$\hat{\mathbf{J}}^2 = \hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar\hat{J}_z.$$

Answer: a) $[\hat{J}_z, \hat{J}_\pm] = [\hat{J}_z, \hat{J}_x \pm i\hat{J}_y]$

$$= \underbrace{[\hat{J}_z, \hat{J}_x]}_{i\hbar J_y} \pm i \underbrace{[\hat{J}_z, \hat{J}_y]}_{-i\hbar J_x}$$

$$= i\hbar\hat{J}_y \pm \hbar J_x = \pm\hbar[\hat{J}_x \pm i\hat{J}_y] = \pm\hbar\hat{J}_\pm$$

b) $\hat{J}_-\hat{J}_+ = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y)$

$$= \hat{J}_x^2 - i\hat{J}_y\hat{J}_x + iJ_xJ_y + \hat{J}_y^2$$

$$= \hat{J}_x^2 + \hat{J}_y^2 + i\underbrace{[J_x, J_y]}_{= i\hbar J_z}$$

$$= \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z$$

$$\Rightarrow \hat{J}^2 = \hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar\hat{J}_z$$

These raising and lowering operators will allow us to jump between eigenstates with

* same value for $\vec{J}^2$
* increments in values for $\hat{J}_z$

Start with eigenstate $|j,m\rangle$
$$\hat{\vec{J}}^2|j,m\rangle = \hbar^2 a_j(a_j+1)|j,m\rangle$$
$$\hat{J}_z|j,m\rangle = \hbar B_m|j,m\rangle$$

$\longrightarrow$ Create new state $\vec{J}_+|j,m\rangle$ $\longrightarrow$ eigenstate of
$$\hat{\vec{J}}^2 \rightsquigarrow \hbar^2 a_j(a_j+1)$$
$$\hat{J}_z \rightsquigarrow \hbar(B_m+1)$$

$\hookrightarrow$ Create new state $\hat{J}_-|j,m\rangle$ $\longrightarrow$ eigenstate of
$$\hat{\vec{J}}^2 \rightsquigarrow \hbar^2 a_j(a_j+1)$$
$$\hat{J}_z \rightsquigarrow \hbar(B_m-1)$$

So $\hat{J}_+$ raises J_z eigenvalue by $+\hbar$
$\hat{J}_-$ lowers J_z " " $-\hbar$

These stem from:

Suppose $|j,m\rangle$ is an eigenstate of $\hat{\vec{J}}^2$ and $\hat{J}_z$:
$$\hat{\vec{J}}^2|j,m\rangle = \hbar^2 a_j(a_j+1)|j,m\rangle$$
$$\hat{J}_z|j,m\rangle = \hbar B_m|j,m\rangle$$

Then $|\Phi\rangle = \hat{J}_+|j,m\rangle$ is also an eigenstate
$$\hat{\vec{J}}^2|\Phi\rangle = \hbar^2 a_j(a_j+1)|\Phi\rangle$$
$$\hat{J}_z|\Phi\rangle = \hbar(B_m+1)|\Phi\rangle$$

Then $|\Psi\rangle = \hat{J}_-|j,m\rangle$ is also an eigenstate:
$$\hat{\vec{J}}^2|\Psi\rangle = \hbar^2 a_j(a_j+1)|\Psi\rangle$$
$$\hat{J}_z|\Psi\rangle = \hbar(B_m-1)|\Psi\rangle$$

Proof: $\hat{\vec{J}}^2|\Phi\rangle = \hat{\vec{J}}^2\hat{J}_+|j,m\rangle$

But $\hat{\vec{J}}^2\hat{J}_+ = \hat{J}_+\hat{\vec{J}}^2$

$\Rightarrow \hat{\vec{J}}^2|\Phi\rangle = \hat{J}_+\hat{\vec{J}}^2|j,m\rangle$

$= \hat{J}_+ \hbar^2 a_j(a_j+1)|j,m\rangle$

$= \hbar^2 a_j(a_j+1)\underbrace{\hat{J}_+|j,m\rangle}_{|\Phi\rangle} = \hbar^2 a_j(a_j+1)|\Phi\rangle.$

$\Rightarrow |\Phi\rangle$ eigenstate.

Then

$\hat{J}_z|\Phi\rangle = \hat{J}_z\hat{J}_+|j,m\rangle$

But $[\hat{J}_z, \hat{J}_+] = +\hbar\hat{J}_+$

$\Rightarrow \hat{J}_z\hat{J}_+ - \hat{J}_+\hat{J}_z = \hbar\hat{J}_+$

$\Rightarrow \hat{J}_z\hat{J}_+ = \hat{J}_+J_z + \hbar\hat{J}_+$

$\Rightarrow \hat{J}_z|\Phi\rangle = \hat{J}_+\underbrace{J_z|j,m\rangle}_{\hbar B_m|j,m\rangle} + \hbar J_+|j,m\rangle$

$= \hbar(B_m+1)\underbrace{\hat{J}_z|j,m\rangle}_{|\Phi\rangle}$ $\boxtimes$

This proves the result.

Thus we can establish a "ladder" of eigenstates. Suppose that j is given.

$\Downarrow$

Fixes a_j

$\Downarrow$

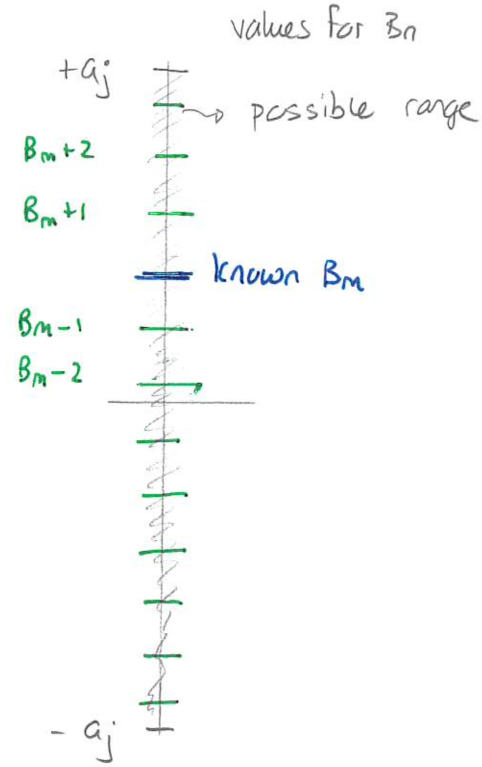
Fixes bounds on B_m

$\Downarrow$

Starting with known B_m we can produce a list $B_m+1, B_m+2, \ldots$

$B_m-1, B_m-2, \ldots$

of other eigenvalues



There are clearly only a finite number in each such ladder. We aim to show that

$\boxed{\text{Every ladder for } B_m \text{ terminates in exactly } \pm a_j \text{ at either end}}$

1) maximum J_z. Suppose that m_{max} is such that $B_{m_{max}}$ is the largest value of B_m in a given ladder. Then consider $|j, m_{max}\rangle$. Then

$$\hat{J}_z \hat{J}_+ |j, m_{max}\rangle = \hbar (B_{m_{max}}+1) \hat{J}_+ |j, m_{max}\rangle$$

would give a greater eigenvalue unless $J_+ |j, m_{max}\rangle = 0$. So

$$\hat{J}_+ |j, m_{max}\rangle = 0$$

2) minimum J_z ~ There is m_{min} such that $|j, m_{min}\rangle$ has the lowest eigenvalue of J_z. So

$$\hat{J}_- |j, m_{min} = 0\rangle.$$

It follows that:

$$\boxed{\begin{aligned} B_{m_{min}} &= -a_j \\ B_{m_{max}} &= a_j \end{aligned}}$$

<u>Proof:</u> $\hat{\vec{J}}^2 |j, m_{max}\rangle = \{\hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar \hat{J}_z\} |j, m_{max}\rangle$

$$= \overset{0}{\cancel{\hat{J}_-\hat{J}_+ |j, m_{max}\rangle}} + \hat{J}_z^2 |j, m_{max}\rangle + \hbar \hat{J}_z |j, m_{max}\rangle$$

$$\Rightarrow \hbar^2 a_j(a_j+1) = \hbar^2 B_{m_{max}}^2 + \hbar^2 \hat{B}_{m_{max}}$$

$$\Rightarrow a_j(a_j+1) = B_{m_{max}}^2 + B_{m_{max}}$$

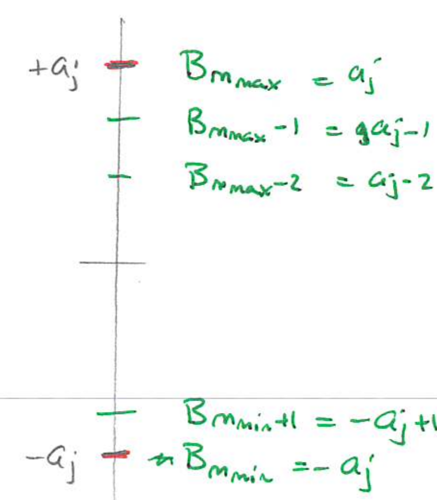
$$\Rightarrow B_{m_{max}}^2 + B_{m_{max}} - a_j(a_j+1) = 0$$

$$\Rightarrow B_{m_{max}} = \frac{-1 \pm \sqrt{1 + 4a_j(a_j+1)}}{2} = \frac{-1 \pm \sqrt{(2a_j+1)^2}}{2}$$

$$\Rightarrow B_{m_{max}} = \underbrace{\frac{-1-2a_j-1}{2}}_{-a_j-1} \quad \text{or} \quad B_{m_{max}} = \underbrace{\frac{-1+2a_j+1}{2}}_{a_j}$$

But $B_{m_{max}}$ cannot be less than a_j. Thus $B_{m_{max}} = a_j$
A similar derivation gives the result for $B_{m_{min}}$. ∎

Thus the maximum and minimum on any such ladder is exactly $\pm a_j$. Thus all ladders are the same. Then the ladder has an integral number of steps, denoted N. So



$$a_j - (-a_j) = N$$

$$\Rightarrow a_j = N/2$$

This restricts possible values of a_j to integers and half integers.

Thus we define

$$j = \frac{N}{2}$$ ~ half the number of ladder steps

$$a_j = j$$

Possible values of j are

$$j = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\ldots$$

Possible values of B_m are

$$B_m = m = -j, -j+1, \ldots j-1, j$$

Thus

The possible eigenstates and eigenvalues of angular momentum are $|j,m\rangle$:

$$\hat{\vec{J}}^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle$$

$$\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle.$$

where possible values for j are

$$j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$$

and, for any given j, possible values of m are

$$m = -j, -j+1, -j+2, \ldots j-2, j-1, j$$

2 Angular momentum eigenstates

For each of the following list all angular momentum eigenstates and eigenvalues: $j = 1/2, j = 1, j = 3/2$, and $j = 2$.

Answer:

For $j = 1/2$ $\leadsto$ eigenvalue $\vec{J}^2 = \hbar^2 \frac{1}{2}\frac{3}{2} = \frac{3}{4}\hbar^2$

possible $m = -\frac{1}{2}, \frac{1}{2}$

States are $|\frac{1}{2}, -\frac{1}{2}\rangle, |\frac{1}{2}, \frac{1}{2}\rangle$

For $j = 1$ $\leadsto$ eigenvalue $\vec{J}^2 = \hbar^2 2$

possible m are $m = -1, 0, 1$

states are $|1,-1\rangle, |1,0\rangle, |1,1\rangle$

For $j = \frac{3}{2}$ $\leadsto$ eigenvalue $\vec{J}^2 = \hbar^2 \frac{3}{2}\frac{5}{2} = \frac{15}{4}\hbar^2$

possible $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

states $|\frac{3}{2}, -\frac{3}{2}\rangle$ $|\frac{3}{2}, -\frac{1}{2}\rangle$ $|\frac{3}{2}, \frac{1}{2}\rangle$ $|\frac{3}{2}, \frac{3}{2}\rangle$

For $j = 2$ $\leadsto$ eigenvalue $\vec{J}^2 = \hbar^2 6$

possible $m = -2, -1, 0, 1, 2$

states $|2,-2\rangle$ $|2,-1\rangle$ $|2,0\rangle$ $|2,1\rangle$ $|2,2\rangle$

Raising + lowering operator action

One can show that

$$\boxed{\begin{aligned}\hat{J}_+ |j,m\rangle &= \hbar\sqrt{j(j+1) - m(m+1)}\, |j,m+1\rangle \\ \hat{J}_- |j,m\rangle &= \hbar\sqrt{j(j+1) - m(m-1)}\, |j,m-1\rangle\end{aligned}}$$

This will be useful for determining expectation values of angular momentum since, for example

$$\hat{J}_x = \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right)$$

$$\hat{J}_y = \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right)$$

So

$$\langle J_x \rangle = \langle \Psi | \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right) | \Psi \rangle$$

and if the state $|\Psi\rangle$ is expressed in terms of angular momentum states then this is readily evaluated.

3 Orbital angular momentum

For each of the following states determine: $\langle L_x \rangle$ and $\langle L_z \rangle$.

a) $|\Psi_1\rangle = |1,1\rangle$

b) $|\Psi_2\rangle = |1,0\rangle$

c) $|\Psi_3\rangle = \frac{1}{2}|1,-1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,1\rangle$.

d) How would you describe each such particle in terms of orbits?

Answer: a) $\langle L_z \rangle = \langle 1,1| \hat{L}_z |1,1\rangle$

and $\hat{L}_z |1,1\rangle = \hbar |1,1\rangle$

$\Rightarrow \langle L_z \rangle = \langle 1,1| \hbar |1,1\rangle \Rightarrow$ $\langle L_z \rangle = \hbar$

$$\langle L_x \rangle = \langle 1,1| \hat{L}_x |1,1\rangle$$

$$= \langle 1,1| \frac{1}{2}(\hat{L}_+ + \hat{L}_-) |1,1\rangle$$

$$= \frac{1}{2}\langle 1,1| \underbrace{\hat{L}_+ |1,1\rangle}_{0} + \frac{1}{2}\langle 1,1| \underbrace{\hat{L}_- |1,1\rangle}_{\hbar\sqrt{2}|1,0\rangle}$$

$$= \frac{\sqrt{2}}{2}\hbar \langle 1,1|1,0\rangle = 0 \quad \Rightarrow \quad \langle L_x \rangle = 0$$

b) $\hat{L}_z |1,0\rangle = 0 |1,0\rangle \quad \Rightarrow \quad \langle L_z \rangle = 0$

$$\hat{L}_x |1,0\rangle = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)|1,0\rangle = \frac{1}{2}\hbar\sqrt{2}\,|1,1\rangle + \frac{1}{2}\hbar\sqrt{2}\,|1,-1\rangle$$

$\Rightarrow \langle 1,0| \hat{L}_x |1,0\rangle = 0 \quad \Rightarrow \quad \langle L_x \rangle = 0$

c) $$\hat{L}_z|\Psi\rangle = \hat{L}_z \tfrac{1}{2}|1,-1\rangle + \hat{L}_z \tfrac{1}{\sqrt{2}}|1,0\rangle + \hat{L}_z \tfrac{1}{2}|1,1\rangle$$

$$= -\frac{\hbar}{2}|1,-1\rangle + 0\,|1,0\rangle + \frac{\hbar}{2}|1,1\rangle$$

$$\langle\Psi|\hat{L}_z|\Psi\rangle = -\frac{\hbar}{2}\langle 1,-1|1,-1\rangle + \frac{\hbar}{2}\langle 1,1|1,1\rangle = 0 \quad \Rightarrow \quad \langle L_z\rangle = 0$$

$$\hat{L}_x|\Psi\rangle = \frac{1}{2}\left(\hat{L}_+ + \hat{L}_-\right)|\Psi\rangle$$

$$= \frac{1}{2}(\hat{L}_+ + \hat{L}_-)\left[\frac{1}{2}|1,-1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,1\rangle\right]$$

$$= \frac{1}{4}\hat{L}_+|1,-1\rangle + \frac{1}{2\sqrt{2}}\hat{L}_+|1,0\rangle + \frac{1}{2\sqrt{2}}\hat{L}_-|1,0\rangle + \frac{1}{4}\hat{L}_-|1,1\rangle$$

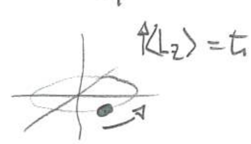
$$= \frac{1}{4}\hbar\sqrt{2}|1,0\rangle + \frac{1}{2\sqrt{2}}\sqrt{2}\hbar|1,1\rangle + \frac{1}{2\sqrt{2}}\sqrt{2}\hbar|1,-1\rangle + \frac{1}{4}\hbar\sqrt{2}|1,0\rangle$$

$$= \frac{\hbar}{\sqrt{2}}|1,0\rangle + \frac{1}{2}\hbar|1,1\rangle + \frac{1}{2}\hbar|1,-1\rangle$$

So $$\langle L_x\rangle = \langle\Psi|\hat{L}_x|\Psi\rangle = \left[\frac{1}{2}\langle 1,-1| + \frac{1}{\sqrt{2}}\langle 1,0| + \frac{1}{2}\langle 1,1|\right]\left[\frac{\hbar}{2}|1,-1\rangle + \frac{\hbar}{\sqrt{2}}|1,0\rangle + \frac{\hbar}{2}|1,1\rangle\right]$$

$$= \frac{\hbar}{4} + \frac{\hbar}{2} + \frac{\hbar}{4} = \hbar$$

d) $|1,1\rangle$ particle orbiting z



$|1,0\rangle$ particle with $\langle L_x\rangle = \langle L_y\rangle = \langle L_z\rangle = 0$

$|\Psi\rangle$ particle with $\langle L_x\rangle = \hbar$ others zero