Phys 321 2023

Lecture 20

Tues: HW 5pm

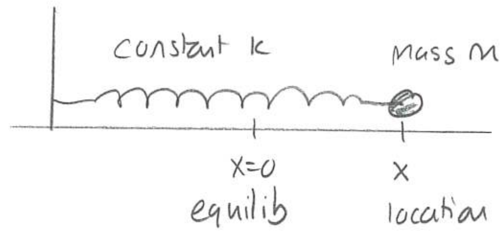
Thurs: Read 9.3 $\rightarrow$ 9.5

Fri: HW 5pm

Harmonic Oscillator

A harmonic oscillator is a system with a restoring force (or a restoring potential energy). We now follow the usual procedure to develop a quantum harmonic oscillator from a classical oscillator.

Consider the classical block/spring.



Newton's 2nd Law gives

$$\vec{F}_{net} = \underbrace{\vec{F}_{spring}}_{-kx} = \underbrace{m\vec{a}}_{m\frac{d^2x}{dt^2}}$$

Thus $m\dfrac{d^2x}{dt^2} = -kx \Rightarrow \dfrac{d^2x}{dt^2} = -\omega^2 x$ where $\boxed{\omega = \sqrt{\frac{k}{m}}}$.

A general solution has the form

$$x(t) = A\cos(\omega t + \phi)$$

and ω is the angular frequency of oscillation.

The total energy of this system is

$$E = K + U_{spring}$$

$$= \frac{1}{2}\frac{p^2}{2m} + \frac{1}{2}kx^2 \qquad \Rightarrow \qquad \boxed{E = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2}$$

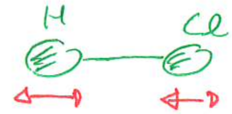
We can therefore describe the classical harmonic oscillator via two parameters:

* system mass, m
* system frequency, ω

We will extend this to quantum oscillators. Examples include:

1) vibrating molecules

— Chem 3D Molecular vibrations

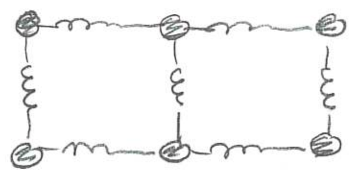


Spectrum?

2) vibrations in crystal lattices

— PhET Normal Modes

- applies to Raman.



3) trapped ions — Innsbruck Page

4) micromechanical oscillators — Schninskii paper

— Aspelmeyer Group, Innsbruck

5) quantum optics

Quantum description

The Hamiltonian for the quantum oscillator is

$$\boxed{\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2}$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators. Then the energy eigenstate $|\phi_E\rangle$ satisfies

$$\hat{H}|\phi_E\rangle = E|\phi_E\rangle$$

In terms of wavefunctions this becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_E}{dx^2} + \frac{1}{2}m\omega^2x^2\,\phi_E(x) = E\,\phi_E(x)$$

which is the TISE for the quantum harmonic oscillator.

3 Simple harmonic oscillator ground state

Consider the candidate energy eigenstate for a simple harmonic oscillator,

$$\phi(x) = Ae^{\alpha x^2}$$

where α is real. Show that this is a solution and determine the value of α and the energy that result in it being a solution.

Answer:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \phi = E\phi$$

$$\frac{d\phi}{dx} = Ae^{\alpha x^2}\frac{d}{dx}(\alpha x^2)$$

$$= 2\alpha x\, Ae^{\alpha x^2} = 2\alpha x \phi$$

$$\frac{d^2\phi}{dx^2} = 2\alpha\phi + 2\alpha x\frac{d\phi}{dx}$$

$$= 2\alpha\phi + (2\alpha x)^2\phi$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[2\alpha + 4\alpha^2x^2\right]\cancel{\phi} + \frac{1}{2}m\omega^2x^2\cancel{\phi} = E\cancel{\phi}$$

$$\Rightarrow x^2\left[-\frac{2\hbar^2\alpha^2}{m} + \frac{1}{2}m\omega^2\right] - \left[\frac{\hbar^2\alpha}{m} + E\right] = 0$$

This can only be true for all x if:

$$\frac{2\hbar^2\alpha^2}{m} = \frac{1}{2}m\omega^2 \Rightarrow \alpha^2 = \frac{m^2\omega^2}{4\hbar^2} \Rightarrow \alpha = \pm\frac{m\omega}{2\hbar}$$

$$E = -\frac{\hbar^2\alpha}{m}$$

Only the negative root can give a normalized function. Thus:

$$\alpha = -\frac{m\omega}{2\hbar} \qquad E = \frac{\hbar\omega}{2} \qquad \phi_E(x) = Ae^{-m\omega x^2/2\hbar}$$

We have found one energy eigenstate

Operator formalism for the harmonic oscillator

A useful alternative approach to determining the properties of a quantum harmonic oscillator uses certain special operators: a creation and an annihilation operator.

First consider the classical energy

$$E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$= \frac{1}{2} m\omega^2 \left\{ \frac{p^2}{m^2\omega^2} + x^2 \right\}$$

$$= \frac{1}{2} m\omega^2 \left(x + \frac{ip}{m\omega} \right)\left(x - \frac{ip}{m\omega} \right)$$

We aim to form the Hamiltonian by a similar factorization method. Then we define:

Annihilation operator:	$\hat{a} := \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right)$
Creation operator:	$\hat{a}^+ := \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right)$

It would then seem that the Hamiltonian should contain a product of these. Possibilities are:

$$\hat{a}^+ \hat{a} \qquad \text{or} \qquad \hat{a}\hat{a}^+$$

Unless $\hat{a}$ and $\hat{a}^+$ commute, the order will be important and will result in different operators. We can show, using

$$[\hat{x}, \hat{p}] = i\hbar \hat{I}$$

that

$$\boxed{[\hat{a}, \hat{a}^+] = \hat{I}}$$

2 Creation and annihilation operator commutation

Show that

$$[\hat{a}, \hat{a}^\dagger] = \hat{I}$$

using

$$[\hat{x}, \hat{p}] = i\hbar\hat{I}.$$

<u>Answer:</u>

$$[\hat{a}, \hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a}$$

$$= \frac{m\omega}{2\hbar}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

$$- \frac{m\omega}{2\hbar}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

$$= \frac{m\omega}{2\hbar}\Big[\cancel{\hat{x}^2} + \frac{i}{m\omega}\left(\hat{p}\hat{x} + \hat{p}\hat{x} - \hat{x}\hat{p} - \hat{x}\hat{p}\right) + \frac{1}{m^2\omega^2}\hat{p}^2$$

$$- \cancel{\hat{x}^2} - \cancel{\frac{1}{m^2\omega^2}\hat{p}^2}\Big]$$

$$= \frac{i}{2\hbar}2\left(\hat{p}\hat{x} - \hat{x}\hat{p}\right) = -\frac{i}{\hbar}\underbrace{\left(\hat{x}\hat{p} - \hat{p}\hat{x}\right)}_{[\hat{x},\hat{p}]}$$

$$= -\frac{i}{\hbar}\,i\hbar\hat{I} = \hat{I} \qquad \boxtimes$$

These give

$$\hat{a}\hat{a}^+ = \hat{a}^+\hat{a} + \hat{I}$$

$$\hat{a}^+\hat{a} = \hat{a}\hat{a}^+ - \hat{I}$$

We can construct the Hamiltonian by inverting the definitions of the creation and annihilation operators. Thus

$$\hat{a} + \hat{a}^{+} = 2\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} = \sqrt{\frac{2m\omega}{\hbar}}\,\hat{x}$$

$$\Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^{+})$$

Similarly

$$\hat{a} - \hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{2i}{m\omega}\right)\hat{p} = i\sqrt{\frac{2}{\hbar m\omega}}\,\hat{p}$$

$$\Rightarrow \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a} - \hat{a}^{+})$$

Thus

$$\boxed{\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^{+}) \\ \hat{p} &= -i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a} - \hat{a}^{+}) \end{aligned}}$$

Then we substitute these into the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

to get.

> The Hamiltonian for a quantum harmonic oscillator is
>
> $$\hat{H} = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\hat{I}\right)$$
>
> where $\hat{a}$ and $\hat{a}^{+}$ satisfy
>
> $$[\hat{a}, \hat{a}^{+}] = \hat{I}$$
>
> and ω is the angular frequency of oscillation.

3 Quantum harmonic oscillator Hamiltonian

Using

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \text{ and}$$
$$\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a} - \hat{a}^\dagger\right),$$

Show that

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{I}\right).$$

Answer:

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega\,\hat{x}^2$$

$$= \frac{1}{2m}\left(-i\sqrt{\frac{m\omega\hbar}{2}}\right)^2\left(\hat{a}-\hat{a}^+\right)^2 + \frac{1}{2}m\omega\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2\left(\hat{a}+\hat{a}^+\right)^2$$

$$= -\frac{1}{2\cancel{m}}\frac{\cancel{m}\omega\hbar}{2}\left(\hat{a}^2 - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} + \hat{a}^{+2}\right)$$

$$+ \frac{1}{2}\cancel{m}\omega^{\cancel{2}}\frac{\hbar}{2\cancel{m}\cancel{\omega}}\left(\hat{a}^2 + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^{+2}\right)$$

$$= \frac{\hbar\omega}{4}\left(-\cancel{\hat{a}^2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} - \cancel{\hat{a}^{+2}}\right.$$
$$\left. + \cancel{\hat{a}^2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \cancel{\hat{a}^{+2}}\right)$$

$$= \frac{\hbar\omega}{2}\left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right)$$

$$= \frac{\hbar\omega}{2}\left(\hat{a}^+\hat{a} + \hat{I} + \hat{a}^+\hat{a}\right)$$

$$= \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\hat{I}\right)$$

Wavefunction representation of creation and annihilation operators

The operators can be represented in terms of their actions on wavefunctions. Thus if

$$|\Psi\rangle \rightsquigarrow \Psi(x)$$

then

$$\hat{a}|\Psi\rangle \rightsquigarrow \underbrace{??}_{\text{some operation}}\ \Psi(x)$$

But

$$\hat{a}|\Psi\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)|\Psi\rangle$$

$$\rightsquigarrow \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\Psi(x)$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\Psi(x)$$

Thus

$$\hat{a}|\Psi\rangle \rightsquigarrow \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\Psi(x)$$

$$\hat{a}^{+}|\Psi\rangle \rightsquigarrow \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\Psi(x)$$

So

$$\hat{a}^{+}\hat{a} \rightsquigarrow \frac{m\omega}{2\hbar}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right) = \frac{m\omega}{2\hbar}\left(x^2 - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}(x\ldots) + x\frac{\hbar}{m\omega}\frac{\partial}{\partial x} - \frac{\hbar^2}{m^2\omega^2}\frac{\partial^2}{\partial x^2}\right)$$

$$= \frac{m\omega}{2\hbar}\left(x^2 - \frac{\hbar}{m\omega} - \cancel{x\frac{\hbar}{m\omega}\frac{\partial}{\partial x}} + \cancel{x\frac{\hbar}{m\omega}\frac{\partial}{\partial x}} - \frac{\hbar^2}{m^2\omega^2}\frac{\partial^2}{\partial x^2}\right)$$

$$= \frac{m\omega}{2\hbar}\left(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{\partial^2}{\partial x^2} - \frac{\hbar}{m\omega}\right)$$

Thus

$$\hat{a}^{+}\hat{a} + \frac{1}{2}\hat{I} \rightsquigarrow \frac{m\omega}{2\hbar}\left[x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{\partial^2}{\partial x^2}\right] - \frac{1}{2} + \frac{1}{2}$$

$$\Rightarrow \hat{H} = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\hat{I}\right)$$

$$\rightsquigarrow \frac{1}{2}m\omega^2\left(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{\partial^2}{\partial x^2}\right)$$

$$\Rightarrow \boxed{\hat{H} \rightsquigarrow \frac{1}{2}m\omega^2 x^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}}$$

This is the position representation of the Hamiltonian.

Harmonic oscillator energy spectrum

We now aim to use the creation and annihilation formalism to determine the energy eigenstates and eigenvalues for the quantum harmonic oscillator. Recall that we seek $|\phi\rangle$ s.t.

$$\hat{H}|\phi\rangle = E|\phi\rangle$$

$$\Rightarrow \hbar\omega\left(\hat{a}^{+}\hat{a} + \tfrac{1}{2}\hat{I}\right)|\phi\rangle = E|\phi\rangle.$$

The scheme is:

Show that, in general

$$E \geqslant \hbar\omega/2$$

Identify a state for which

$$E = \tfrac{1}{2}\hbar\omega$$

and show that this is unique

Show that if $|\phi\rangle$ is an energy eigenstate with energy E then

$$\hat{a}|\phi\rangle$$

is an eigenstate with energy $E-\hbar\omega$

and

$$\hat{a}^{+}|\phi\rangle$$

is an eigenstate with energy $E+\hbar\omega$

Establish a ladder of states

$\vdots$

$|3\rangle$ —— $E = 7\hbar\omega/2$

$|2\rangle$ —— $E = 5\hbar\omega/2$

$|1\rangle$ —— $E = \dfrac{3\hbar\omega}{2}$

$|0\rangle$ —— $E = \dfrac{\hbar\omega}{2}$

Show that there is only one ladder

<u>First result</u>. For any energy eigenstate, the eigenvalue E satisfies

$$\boxed{E \geqslant \hbar\omega/2}$$

<u>Proof</u>: The eigenvalue equation is

$$\hat{H}|\phi\rangle = E|\phi\rangle$$

Then

$$\langle\phi|\hat{H}|\phi\rangle = E\underbrace{\langle\phi|\phi\rangle}_{=1} \quad \Rightarrow \quad E = \langle\phi|\hat{H}|\phi\rangle.$$

But $\hat{H} = \hbar\omega\left[\hat{a}^{+}\hat{a} + \frac{1}{2}\hat{I}\right]$ gives

$$E = \hbar\omega\left[\langle\phi|\hat{a}^{+}\hat{a}|\phi\rangle + \frac{1}{2}\underbrace{\langle\phi|\phi\rangle}_{1}\right]$$

$$= \hbar\omega\left[\frac{1}{2} + \langle\phi|\hat{a}^{+}\hat{a}|\phi\rangle\right]$$

Then conside $\langle\phi|\hat{a}^{+}\hat{a}|\phi\rangle$. Let $|\psi\rangle = \hat{a}|\phi\rangle$. Then

$$\langle\psi| = |\psi\rangle^{+} = \left(\hat{a}|\phi\rangle\right)^{+} = \langle\phi|\hat{a}^{+}$$

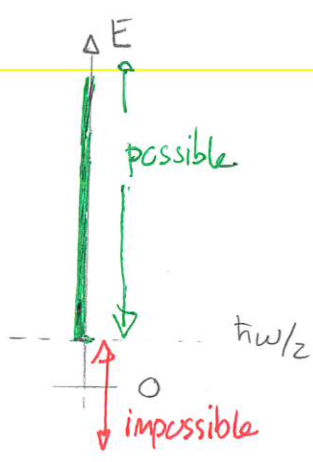
Thus

$$\langle\phi|\hat{a}^{+}\hat{a}|\phi\rangle = \langle\psi|\psi\rangle. \rightsquigarrow \text{real number.}$$

But the inner product of a state with itself cannot be negative. Thus $\langle\phi|\hat{a}^{+}\hat{a}|\phi\rangle \geqslant 0$. Thus

$$E \geqslant \hbar\omega/2. \qquad \blacksquare$$

This restricts the energy range.



Second: If $|\phi\rangle$ is an eigenstate with energy E then

$$\boxed{\begin{array}{l} \hat{a}|\phi\rangle \text{ is an eigenstate with energy } E+\hbar\omega \\ \hat{a}^+|\phi\rangle \text{ is an eigenstate with energy } E-\hbar\omega \\ \hat{a}\,|\phi\rangle = 0 \quad \text{if} \quad E=\hbar\omega/2 \end{array}}$$

Proof: We will first show an important rule.

$$[\hat{H},\hat{a}] = -\hbar\omega\,\hat{a}$$

$$[\hat{H},\hat{a}^+] = \hbar\omega\hat{a}^+$$

The first follows from

$$[\hat{H},\hat{a}] = [\hbar\omega(\hat{a}^+\hat{a}+\tfrac{1}{2}\hat{I}),\hat{a}] = \hbar\omega\{[\hat{a}^+\hat{a},\hat{a}] + \tfrac{1}{2}\cancel{[\hat{I},\hat{a}]}\}$$

$$= \hbar\omega[\hat{a}^+\hat{a},\hat{a}]$$

Then

$$[\hat{a}^+\hat{a},\hat{a}] = \hat{a}^+\hat{a}\hat{a} - \underbrace{\hat{a}\hat{a}^+}_{\hat{a}^+\hat{a}+\hat{I}}\hat{a}$$

$$= \cancel{\hat{a}^+\hat{a}\hat{a}} - \cancel{\hat{a}^+\hat{a}\hat{a}} - \hat{a}$$

$$\Rightarrow [\hat{H},\hat{a}] = -\hbar\omega\hat{a}$$

A similar derivation could give the other result. Or

$$\underbrace{[\hat{H},\hat{a}]^+}_{} = -\hbar\omega\hat{a}^+$$

$$[\hat{a}^+,\hat{H}] = -\hbar\omega\hat{a}^+$$

$$-[\hat{H},\hat{a}^+] = -\hbar\omega\hat{a}^+ \quad \Rightarrow \quad [\hat{H},\hat{a}^+] = \hbar\omega\hat{a}^+$$

Now consider

$$[\hat{H},\hat{a}]|\phi\rangle = -\hbar\omega\,\hat{a}|\phi\rangle$$

$$\Rightarrow (\hat{H}\hat{a}-\hat{a}\hat{H})|\phi\rangle = -\hbar\omega\hat{a}|\phi\rangle$$

$$\Rightarrow \hat{H}\hat{a}|\phi\rangle - \hat{a}E|\phi\rangle = -\hbar\omega\hat{a}|\phi\rangle$$

$$\Rightarrow \hat{H}\hat{a}|\phi\rangle - E\hat{a}|\phi\rangle = -\hbar\omega\hat{a}|\phi\rangle$$

$$\Rightarrow \hat{H}\hat{a}|\phi\rangle = E\hat{a}|\phi\rangle - \hbar\omega\hat{a}|\phi\rangle.$$

$$\Rightarrow \hat{H}\underbrace{\hat{a}|\phi\rangle}_{\substack{\text{state}\\|\psi\rangle}} = (E-\hbar\omega)\underbrace{\hat{a}|\phi\rangle}_{\substack{\text{state}\\|\psi\rangle}} \qquad \Rightarrow \qquad \hat{H}|\psi\rangle = (E-\hbar\omega)|\psi\rangle$$

Thus $\hat{a}|\phi\rangle$ is an eigenstate of $\hat{H}$ with energy $E-\hbar\omega$.
The result for $\hat{a}^{+}|\phi\rangle$ follows in the same way.

Now suppose that $E=\hbar\omega/2$. Then

$$\hat{H}|\phi\rangle = \frac{\hbar\omega}{2}|\phi\rangle$$

$$\Rightarrow \hbar\omega(\hat{a}^{+}\hat{a}+\tfrac{1}{2}\hat{I})|\phi\rangle = \frac{\hbar\omega}{2}|\phi\rangle.$$

$$\Rightarrow \hat{a}^{+}\hat{a}|\phi\rangle + \tfrac{1}{2}|\phi\rangle = \tfrac{1}{2}|\phi\rangle$$

$$\Rightarrow \hat{a}^{+}\hat{a}|\phi\rangle = 0$$

$$\Rightarrow \langle\phi|\hat{a}^{+}\hat{a}|\phi\rangle = 0$$

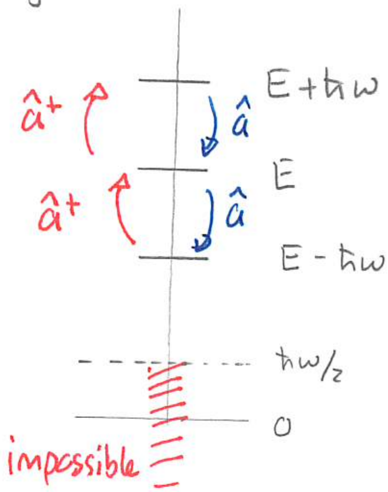
The inner product of $\hat{a}|\phi\rangle$ with itself is zero. This is only possible if

$$\hat{a}|\phi\rangle = 0.$$

Also if $\hat{a}|\phi\rangle=0$ then $|\phi\rangle$ is an energy eigenstate with eigenvalue $\hbar\omega/2$ since

$$\hat{H}|\phi\rangle = \hbar\omega(\hat{a}^{+}\hat{a}+\tfrac{1}{2}\hat{I})|\phi\rangle = \hbar\omega[\hat{a}^{+}\overset{0}{\cancel{\hat{a}|\phi\rangle}}+\tfrac{1}{2}|\phi\rangle] = \frac{\hbar\omega}{2}|\phi\rangle$$

This establishes a ladder of possible energies and states. The "rungs" on the ladder are separated by jumps of $\hbar\omega$. Thus



there will be infinitely many states on the ladder. The lowest rung on the ladder is above $\hbar\omega/2$.

We can see that the lowest rung should have energy

$$\frac{\hbar\omega}{2} \leq E < \frac{\hbar\omega}{2} + \hbar\omega. \quad = \frac{3\hbar\omega}{2}$$

We can show that, unless $E = \frac{\hbar\omega}{2}$ for the lowest rung, then there will be a contradiction. Suppose that

$$\frac{\hbar\omega}{2} < E < \frac{3\hbar\omega}{2}$$ for state $|\phi_0\rangle$. Then consider $\hat{a}\,|\phi_0\rangle$.

This is an eigenstate of $\hat{H}$ with eigenvalue $E' = E - \hbar\omega$. But

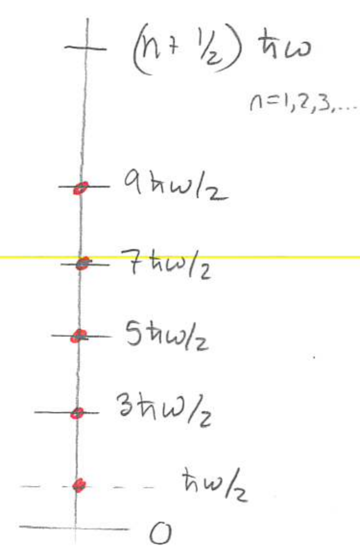
$$E' < \frac{\hbar\omega}{2}$$

and this is impossible, unless perhaps $\hat{a}|\phi_0\rangle = 0$. But in that case $|\phi_0\rangle$ is an eigenstate with energy $\hbar\omega/2$. Thus

$\boxed{\text{The lowest energy in any ladder is } \hbar\omega/2}$

This fixes the ladder. There may still be multiple states at each level. We show that

1) there is only one grand state
2) this implies there is only one state at each other level.



Thus we get

The state corresponding to wavefunction

$$\phi_0(x) = C_0 e^{-m\omega x^2/2\hbar} \sim |0\rangle$$

is the <u>unique</u> eigenstate with energy $\hbar\omega/2$.

<u>Proof</u>: We have already seen that $\phi_0(x)$ satisfies the TISE. with $E = \hbar\omega/2$. This establishes the existence of the state.

Suppose that another such state exists. Denote this $\phi_0'(x)$. Then

$$\phi_0'(x) = f\,\phi(x)$$

for some $f = \phi_0'(x)/\phi(x)$. Now

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_0'}{dx} + \frac{1}{2}m\omega^2 x^2 \phi_0' = \frac{\hbar\omega}{2}\phi_0'$$

$$\frac{d\phi_0'}{dx} = f\frac{d\phi_0}{dx} + \frac{df}{dx}\phi_0$$

$$\frac{d^2\phi_0'}{dx^2} = f\frac{d^2\phi_0}{dx^2} + 2\frac{df}{dx}\frac{d\phi_0}{dx} + \frac{d^2f}{dx^2}\phi_0$$

$$-\frac{\hbar^2}{2m}\left[f\frac{d^2\phi_0}{dx^2} + 2\frac{df}{dx}\frac{d\phi_0}{dx} + \frac{d^2f}{dx^2}\phi_0\right] + \frac{1}{2}m\omega^2x^2 f\phi_0 = \frac{\hbar\omega}{2}f\phi_0$$

$$f\underbrace{\left[-\frac{\hbar^2}{2m}\frac{d^2\phi_0}{dx^2} + \frac{1}{2}m\omega^2x^2\phi_0\right]}_{\frac{\hbar\omega}{2}\phi_0} - \frac{\hbar^2}{2m}\left[2\frac{df}{dx}\frac{d\phi_0}{dx} + \frac{d^2f}{dx^2}\phi_0\right] = \frac{\hbar\omega}{2}f\phi_0$$

$$\Rightarrow \quad 2\frac{df}{dx}\frac{d\phi_0}{dx} + \frac{d^2f}{dx}\phi_0 = 0$$

But

$$\frac{d\phi_0}{dx} = -\frac{m\omega}{\hbar} x \phi_0$$

gives:

$$-\frac{2m\omega}{\hbar} x \frac{df}{dx} + \frac{d^2 f}{dx} = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{2m\omega}{\hbar} x \frac{df}{dx} \quad \Rightarrow \quad \frac{df}{dx} = A e^{m\omega x^2/\hbar}$$

for a constant A. Now

$$\frac{d\phi_0'}{dx} \rightarrow \text{finite as } x \rightarrow \infty$$

Then:

$$\phi_0'(x) = A e^{m\omega x^2/\hbar} \underbrace{\phi_0(x)}_{\text{const } e^{-m\omega x^2/2\hbar}} \quad \rightarrow \quad \text{Const } A e^{m\omega x^2/2\hbar}$$

Thus $A = 0$ for this to be band. So

$$\frac{df}{dx} = 0 \Rightarrow f = \text{constant}.$$

so $\phi_0'(x)$ is a multiple of $\phi(x)$. This establishes uniqueness.

Now suppose that there are two eigenstates with energy $\frac{\hbar\omega}{2}+\hbar\omega$.

Denote these

$$|\phi_1\rangle, |\phi_1'\rangle. \quad \Rightarrow \quad \hbar\omega\left(\hat{a}^+\hat{a}+\tfrac{1}{2}\hat{I}\right)|\phi_1\rangle = \frac{3\hbar\omega}{2}|\phi_1\rangle.$$

$$\Rightarrow \quad \hat{a}^+\hat{a}|\phi_1\rangle = |\phi_1\rangle.$$

Then

$$\hat{a}|\phi_1\rangle, \quad \hat{a}|\phi_1'\rangle$$

both have energy $\hbar\omega/2$. They are equal. So

$$\hat{a}|\phi_1\rangle = \hat{a}|\phi_1'\rangle.$$

Then

$$\underbrace{\hat{a}^+\hat{a}|\phi_1\rangle}_{|\phi_1\rangle} = \underbrace{\hat{a}^+ a|\phi_1'\rangle}_{|\phi_1'\rangle}.$$

Thus there is only one state with energy $\hbar\omega/2+\hbar\omega$. This continues upward through the energies. Thus

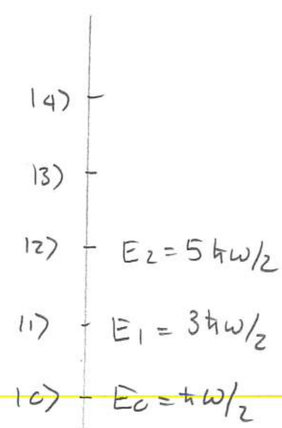
For the harmonic oscillator there is a single "ladder" of energy eigenstates with exactly one state at each rung.

The states are indexed with

$n=0,1,2,\ldots$

and are denoted

$|n\rangle$.



The associated energy is

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$