Phys 321 2023

Lecture 18

Tues: HW 5pm

Thurs: Seminar

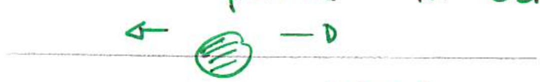
Read text 6.1 - 6.2
notes.

Fri: HW 5pm

Momentum measurement statistics, momentum representation

The state of any particle in one dimension can be translated into a wavefunction that depends on position and time and can be used to describe the statistics of position measurement outcomes. This entire scheme can be adapted to describe the outcomes of momentum measurements, starting from the same state.

particle in state



$|\Psi(t)\rangle$

Position measurements

What is probability that position measurement gives outcome $x_a \leq x \leq x_b$?

Position space wave function
$\Psi(x,t)$

position

Position probability density
$P(x,t) = |\Psi(x,t)|^2$

$$\mathrm{Prob}(x_a \leq x \leq x_b) = \int_{x_a}^{x_b} P(x,t)\,dx$$

Momentum measurements

What is probability that momentum measurement gives outcome $p_a \leq p \leq p_b$

Momentum space wavefunction
$\tilde{\Psi}(p,t)$

momentum

Momentum probability density
$\tilde{P}(p,t) = |\tilde{\Psi}(p,t)|^2$

$$\mathrm{Prob}(p_a \leq p \leq p_b) = \int_{p_a}^{p_b} \tilde{P}(p,t)\,dp$$

The framework is the same but the different types of questions use different wavefunctions. However, these are derived from the same state $|\Psi(t)\rangle$. Thus they must be related. So

* Given $\Psi(x,t)$ how can we determine $\tilde{\Psi}(p,t)$?
* Given $\tilde{\Psi}(p,t)$ how can we determine $\Psi(x,t)$?

We can use formal manipulations involving position and momentum states. To recap:

Position states	Momentum states
$\|x\rangle$ any real x	$\|p\rangle$ any real p
measure position and get x with certainty	measure momentum and get p with certainty.

Then by the usual rules:

$$\Psi(x,t) = \langle x|\Psi(t)\rangle \qquad \text{and} \qquad \tilde{\Psi}(p,t) = \langle p|\Psi(t)\rangle.$$

Thus

$$\begin{aligned}\tilde{\Psi}(p,t) = \langle p|\Psi(t)\rangle &= \langle p|\hat{I}|\Psi(t)\rangle \\ &= \langle p|\int_{-\infty}^{\infty} |x\rangle\langle x|\,dx\,|\Psi(t)\rangle \\ &= \int_{-\infty}^{\infty} \langle p|x\rangle \underbrace{\langle x|\Psi(t)\rangle}_{\Psi(x,t)}\,dx\end{aligned}$$

Now $\langle p|x\rangle = \langle x|p\rangle^*$ and thus

$$\tilde{\Psi}(p,t) = \int_{-\infty}^{\infty} \langle x|p\rangle^* \,\Psi(x,t)\,dx$$

Thus, to translate between the two we just need

$\langle x|p\rangle$ = position wavefunction for a momentum state.

$= \Psi_p(x)$

Now the momentum state satisfies

$$\hat{p}|p\rangle = p|p\rangle$$

where $\hat{p}$ is the momentum operator. In terms of wavefunctions this translates to:

$$-i\hbar\frac{\partial}{\partial x}\Psi_p(x) = p\,\Psi_p(x)$$

and the solution to this is

$$\Psi_p(x) = A e^{ipx/\hbar}$$

where A is a constant. We choose $A = \frac{1}{\sqrt{2\pi\hbar}}$. Thus

$$\Psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar} = \langle x|p\rangle.$$

Thus $\langle p|x\rangle = \langle x|p\rangle^* = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-ipx/\hbar}$. This leaves:

Given a state $|\Psi(t)\rangle$ with position wavefunction $\Psi(x,t)$ the associated momentum wavefunction is

$$\tilde{\Psi}(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\Psi(x,t)\,dx$$

Conversely

Given a state $|\Psi(t)\rangle$ with momentum wavefunction $\tilde{\Psi}(p,t)$ the associated position wavefunction is:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{ipx/\hbar}\,\tilde{\Psi}(p,t)\,dp.$$

1 Momentum wavefunction for a sharply localized particle

Suppose that the position wavefunction for a particle at one instant is

$$\Psi(x) = \begin{cases} A & \text{if } -L/2 \leqslant x \leqslant L/2 \\ 0 & \text{otherwise} \end{cases}$$

where $L > 0$.

a) Determine the constant A.

b) How would you describe the location of the particle as represented by this wavefunction.

c) Determine the momentum space wavefunction.

d) Determine an expression for the momentum space probability density. How would you describe the momentum of this particle?

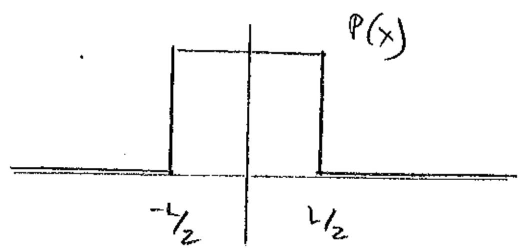
a) $$1 = \int_{-\infty}^{\infty} |\Psi(x)|^2 \, dx = A^2 \int_{-L/2}^{L/2} dx = A^2 L \Rightarrow A = \frac{1}{\sqrt{L}}$$

Thus

$$\Psi(x) = \begin{cases} \frac{1}{\sqrt{L}} & -\frac{L}{2} \leq x \leq \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

b) The probability density is

$$P(x) = |\Psi(x)|^2 = \begin{cases} \frac{1}{L} & -\frac{L}{2} \leq x \leq \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$



The particle is equally likely to be found anywhere in the range $-L/2 \leq x \leq L/2$

c) $$\tilde{\Psi} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \, \Psi(x) dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-L/2}^{L/2} \frac{1}{\sqrt{L}} e^{-ipx/\hbar} \, dx$$

Thus $$\tilde{\Psi}(p) = \frac{1}{\sqrt{2\pi\hbar L}}\left(\frac{-\hbar}{ip}\right) e^{-ipx/\hbar}\Big|_{-L/2}^{L/2}$$

$$= \sqrt{\frac{\hbar}{2\pi L}}\,\frac{i}{p}\underbrace{\left[e^{-ipL/2\hbar} - e^{ipL/2\hbar}\right]}_{=-2i\sin\left(\frac{pL}{2\hbar}\right)}$$

$$= \sqrt{\frac{\hbar}{2\pi L}}\,\frac{2}{p}\sin\left(\frac{pL}{2\hbar}\right)$$

$$\Rightarrow \quad \tilde{\Psi}(p) = \sqrt{\frac{2\hbar}{\pi L}}\;\frac{\sin\left(\frac{pL}{2\hbar}\right)}{p}$$

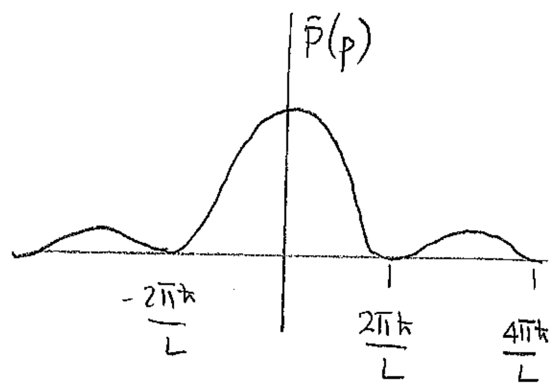
d) $$\tilde{P}(p) = |\tilde{\Psi}(p)|^2 = \frac{2\hbar}{\pi L}\;\frac{\sin^2\left(\frac{pL}{2\hbar}\right)}{p^2}$$

We can plot noting

as $p \to \infty$ $\tilde{P}(p) \to 0$

$p \to 0$ $\tilde{P}(p) \to \frac{2\hbar}{\pi L}\frac{1}{p^2}\left(\frac{pL}{2\hbar}\right)^2$

$$= \frac{2\hbar L^2}{\pi L \hbar^2} = \frac{2L}{\pi\hbar}$$



Then $\tilde{P}(p) = 0$ when

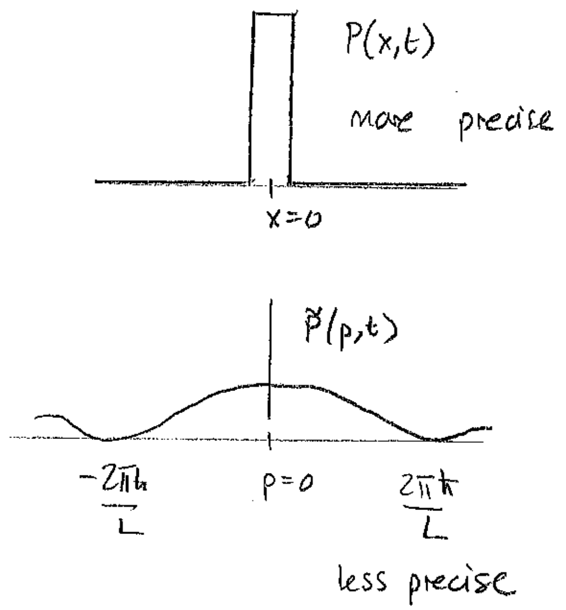
$$\frac{pL}{2\hbar} = n\pi \quad \Rightarrow p = \frac{2\pi\hbar}{L}n$$

for any integer n. A measurement of momentum is overwhelmingly likely to give a result in the range $-\frac{2\pi\hbar}{L} \leq p \leq \frac{2\pi\hbar}{L}$

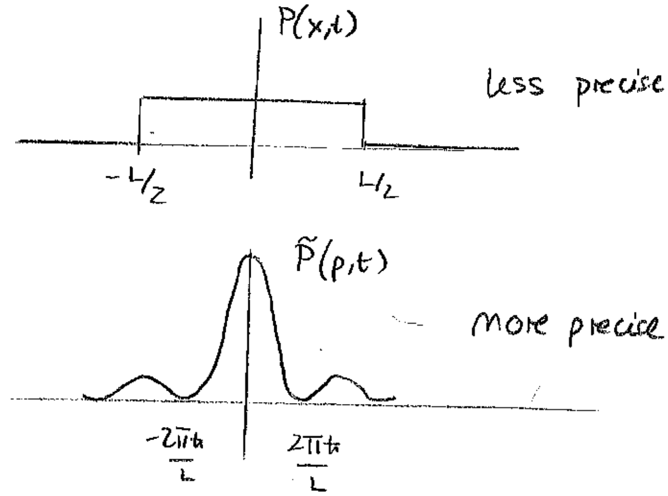
This example illustrates a trade-off

* As L decreases the position becomes more localized but the momentum less localized.
* As L increases the position becomes less localized but the momentum more localized.

Smaller L



Larger L



We can ask

* Is this trade-off true for all states that are represented by wavefunctions?
* Can we quantify the tradeoff?

We will quantify the trade-off in terms of position and momentum uncertainties:

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$$

2 Gaussian wavefunction

Consider the state described by

$$\Psi(x) = \left(\frac{1}{\pi a^2}\right)^{1/4} e^{-(x-x_0)^2/2a^2}.$$

The momentum wavefunction is

$$\tilde{\Psi}(p) = \left(\frac{a^2}{\pi \hbar^2}\right)^{1/4} e^{-ipx_0/\hbar}\, e^{-p^2a^2/2\hbar^2}.$$

a) Determine $\langle x\rangle$ and the uncertainty in position measurement outcomes, Δx.

b) Determine $\langle p\rangle$ and the uncertainty in momentum measurement outcomes Δp.

c) Determine $\Delta x \Delta p$. As the precision of the knowledge about position increases what happens to the knowledge about momentum?

Note the following integrals, all true if the real part of α is positive.

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x+\gamma} \mathrm{d}x = e^{\gamma}\, e^{\beta^2/4\alpha} \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x e^{-\alpha x^2+\beta x+\gamma} \mathrm{d}x = e^{\gamma}\, e^{\beta^2/4\alpha} \frac{\beta}{2} \sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2+\beta x+\gamma} \mathrm{d}x = e^{\gamma}\, e^{\beta^2/4\alpha} \frac{\beta^2+2\alpha}{4} \sqrt{\frac{\pi}{\alpha^5}}.$$

Answer: a) $\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x)|^2 dx$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{1}{\pi a^2}}\; x\, e^{-(x-x_0)^2/a^2}\, dx$$

$$= \frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{\infty} x\, e^{-(x^2-2xx_0+x_0^2)/a^2}\, dx \qquad \alpha = \frac{1}{a^2} \quad \beta = \frac{2x_0}{a^2} \quad \gamma = -x_0^2/a^2$$

$$= \frac{1}{\sqrt{\pi a^2}} \left[e^{-x_0^2/a^2}\, e^{\left(\frac{2x_0}{a^2}\right)^2/4/a^2}\; \frac{x_0}{a^2} \sqrt{\pi a^6} \right]$$

$$= \sqrt{\frac{\pi a^6}{\pi a^2}}\; e^{-x_0^2/a^2} e^{x_0^2/a^2}\; \frac{x_0}{a^2} = x_0$$

$$\Rightarrow \langle x\rangle = \langle x_0\rangle$$

We need $\langle x^2 \rangle$.

$$\langle x^2 \rangle = \frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{a^2} + \frac{2x_0}{a^2}x - \frac{x_0^2}{a^2}} dx$$

$$= \frac{1}{\sqrt{\pi a^2}} e^{-x_0^2/a^2} \int x^2 e^{-\frac{x^2}{a^2} + \frac{2x_0}{a^2}x} dx$$

$$= \frac{1}{\sqrt{\pi a^2}} \cancel{e^{-x_0^2/a^2}}\ \cancel{e^{x_0^2/a^2}} \left[\frac{\left(\frac{2x_0}{a^2}\right)^2 + \frac{2}{a^2}}{4} \right] \sqrt{\frac{\pi a^{10}}{1}}$$

$$= \sqrt{\frac{\cancel{\pi} a^{10}}{\cancel{\pi} a^2}} \frac{1}{4} 2 \left[2\frac{x_0^2}{a^4} + \frac{1}{a^2} \right] = \frac{a^4}{2} \left[\frac{2x_0^2}{a^4} + \frac{1}{a^2} \right]$$

$$= x_0^2 + \frac{a^2}{2}$$

Thus $\quad \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\cancel{x_0^2} + \frac{a^2}{2} - \cancel{x_0^2}}$

$$\Rightarrow \Delta x = \frac{a}{\sqrt{2}}$$

c) $\langle p \rangle = \displaystyle\int_{-\infty}^{\infty} p |\Psi(p)|^2 dp = \sqrt{\frac{a^2}{\pi \hbar^2}} \int_{-\infty}^{\infty} p e^{-p^2 a^2/\hbar^2} dp = 0$

$$\langle p^2 \rangle = \sqrt{\frac{a^2}{\pi \hbar^2}} \int_{-\infty}^{\infty} p^2 e^{-p^2 a^2/\hbar^2} dp = \sqrt{\frac{a^2}{\pi \hbar^2}} \cdot \frac{a^2}{2\hbar^2} \sqrt{\frac{\pi \hbar^{10}}{a^{10}}}$$

$$= \frac{a^2}{2\hbar^2} \frac{\hbar^4}{a^4} = \frac{1}{2} \frac{\hbar^2}{a^2}$$

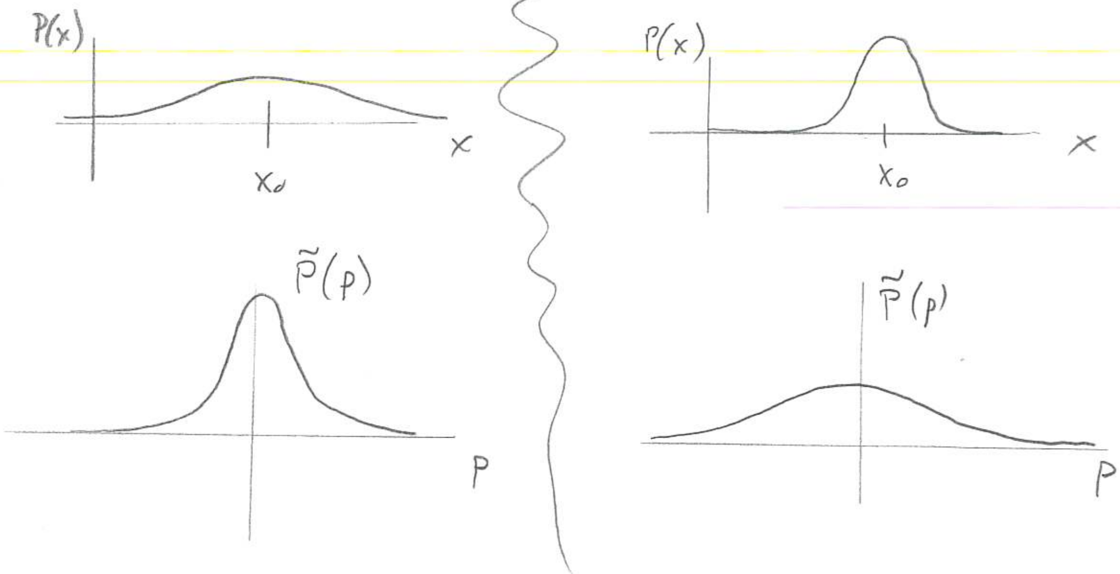
Thus

$$\Delta p = \sqrt{\frac{1}{2}\frac{\hbar^2}{a^2}} = \frac{1}{\sqrt{2}}\frac{\hbar}{a} \qquad \Delta p = \frac{1}{\sqrt{2}}\frac{\hbar}{a}$$

c) $$\Delta x\, \Delta p = \frac{a}{\sqrt{2}}\frac{\hbar}{\sqrt{2}\, a} = \frac{\hbar}{2} \quad \Rightarrow \quad \Delta x \Delta p = \frac{\hbar}{2}$$

As the precision of position increases, the precision of momentum decreases. Again there is a trade-off but this can be quantified precisely in terms of the uncertainties

Larger a — Smaller a



Position-momentum uncertainty relation

Recall that for any state $|\Psi\rangle$ a general rule is for two observable quantities, A and B.

$$\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle \Psi | [\hat{A}, \hat{B}] | \Psi \rangle \right|$$

This is the general uncertainty relation. Evaluating this for position and momentum requires the commutator

$$\boxed{[\hat{x}, \hat{p}] = i\hbar \hat{I}}$$

Proof: Let $|\Psi\rangle$ be any state with position wavefunction $\Psi(x)$.

Then

$$[\hat{x}, \hat{p}] |\Psi\rangle = \hat{x}\hat{p} |\Psi\rangle - \hat{p}\hat{x} |\Psi\rangle$$

corresponds to

$$x \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi(x) - -i\hbar \frac{\partial}{\partial x} \left[x \Psi(x) \right]$$

$$= -i\hbar x \frac{\partial \Psi}{\partial x} + i\hbar \left[\Psi(x) + x \frac{\partial \Psi}{\partial x} \right]$$

$$= i\hbar \Psi(x)$$

Thus

$$[\hat{x}, \hat{p}] |\Psi\rangle = i\hbar \hat{I} |\Psi\rangle$$

for all $|\Psi\rangle$. So

$$[\hat{x}, \hat{p}] = i\hbar \hat{I} \qquad \blacksquare$$

Then, for any state $|\Psi\rangle$

$$\Delta x \Delta p \geq \frac{1}{2} \left| \langle\Psi| [\hat{x}, \hat{p}] |\Psi\rangle \right|$$

$$= \frac{1}{2} \left| \langle\Psi| i\hbar \hat{I} |\Psi\rangle \right|$$

$$= \frac{1}{2} \left| i\hbar \langle\Psi| \hat{I} |\Psi\rangle \right|$$

$$= \frac{1}{2} \left| i\hbar \underbrace{\langle\Psi|\Psi\rangle}_{=1} \right| = \frac{\hbar}{2}$$

Thus, from general considerations we arrive at the Heisenberg uncertainty principle:

$$\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$$

for any state $|\Psi\rangle$.

The previous example showed that the Gaussian wavefunction attains the exact lower bound.

$$\boxed{\text{For} \quad \Psi(x) = A e^{-(x-x_0)^2/2a^2}, \qquad \Delta x \Delta p = \hbar/2}$$

This is the most precise in terms of the product. In general states will not usually attain the lower bound. For example consider

$$\Psi(x) = \sqrt{\frac{2a^3}{\pi x_0}} \frac{1}{(x/x_0)^2 + a^2}$$

where x_0 has units of position and a is dimensionless.

For this state,

$$\Psi(x) \rightsquigarrow \tilde{\Psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x)\, dx$$

$\langle x \rangle = 0$

$\Delta x = x_0 a$ $\qquad\qquad$ $\Delta p = \frac{\hbar}{\sqrt{2}\, x_0 a}$

Then here $\Delta x \Delta p = \frac{\hbar}{\sqrt{2}} = \sqrt{2}\, \frac{\hbar}{2} > \frac{\hbar}{2}$ and this does not attain the lower band.