Phys 321 2023

Lecture 14

Tues: HW 5pm

Thurs: Read Pg 120-131 (My notes
Text 5.3

Fri: HW 5pm

Infinite dimensional quantum systems

A key aspect of the mathematical description of quantum systems is that the set of all kets/states forms a complex vector space and that this space has a dimension. In mathematical terms, the dimension of the vector space is the number of vectors in a basis required to express any state. Thus

spin-$\frac{1}{2}$ $\quad |\Psi\rangle = c_+ \underbrace{|+\hat{z}\rangle}_{} + c_- \underbrace{|-\hat{z}\rangle}_{}$ basis vectors $\rightsquigarrow$ dimension $= 2$

photon in interferometer $\quad |\Psi\rangle = c_0 |0\rangle + c_1 |1\rangle$ basis vectors $\rightsquigarrow$ dimension $= 2$

In physical terms:

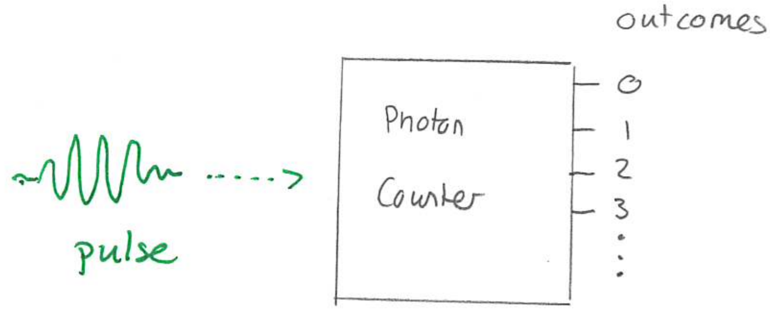
> The dimension of the space of all states equals the maximum number of distinct mutually incompatible outcomes for any single measurement.

The systems that we have considered so far have been two-dimensional. There are other finite-dimensional quantum systems such as spin-1 (three dimensions). We now consider infinite-dimensional systems

As an example consider a pulse of light. We could measure the number of photons in the pulse.

The possible outcomes of this measurement are

$$n = 0, 1, 2, 3, 4, \ldots$$



Clearly there are infinitely many possibilities.

Since these are mutually distinct outcomes of one measurement, the space for all kets will have to be infinite-dimensional. The framework we have developed applies to these systems but:

1) it requires more abstract representation than column vectors
2) we need computational techniques for infinite dimensional sums
3) we need to check convergence of sums.

A generic example, including the light pulse above, considers energy measurements. Suppose that the outcomes and states are as listed. Thus

System in state $|\phi_j\rangle$ $\Rightarrow$ energy measurement gives outcome E_j with certainty.

Outcome	State
E_1	$\vert\phi_1\rangle$
E_2	$\vert\phi_2\rangle$
E_3	$\vert\phi_3\rangle$
$\vdots$	$\vdots$

Infinitely many different possibilities

Then these states must be orthonormal:

$$\langle \phi_n | \phi_n \rangle = 1 \quad \text{any } n$$
$$\langle \phi_m | \phi_n \rangle = 0 \quad \text{if } m \neq n$$

A concise way to describe this uses the Kronecker delta function:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Thus

$$\boxed{\langle \phi_m | \phi_n \rangle = \delta_{mn}}$$

The general state of the system has form.

$$|\Psi\rangle = \sum_n a_n |\phi_n\rangle$$

where a_n are complex. We could attempt to express this as a column vector

$$|\Psi\rangle \rightsquigarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}$$

Then:

$$\langle \Psi | = (|\Psi\rangle)^\dagger = \left(\sum_n a_n |\phi_n\rangle \right)^\dagger$$

$$= \sum_n a_n^* \langle \phi_n |.$$

Thus we have

$$\boxed{|\Psi\rangle = \sum a_n |\phi_n\rangle \Rightarrow \langle \Psi | = \sum_n a_n^* \langle \phi_n |}$$

States of light

As an example consider a system consisting of monochromatic light, i.e. with a single frequency f. Then let $\omega = 2\pi f$. We can measure the number of photons in this pulse. This is a single measurement with outputs and states as listed. Then

Photon number measurement

outcome	state
0	$\|0\rangle$
1	$\|1\rangle$
2	$\|2\rangle$
3	$\|3\rangle$
$\vdots$	$\vdots$
n	$\|n\rangle$
$\vdots$	$\vdots$

$$\langle m|n\rangle = \delta_{mn}$$

and the general state is

$$|\Psi\rangle = \sum a_n |n\rangle.$$

Note that each photon also has definite energy $\hbar\omega$. So these are also the states associated with energy measurements. In this case the Hamiltonian is constructed as

Energy measurement

outcome	state
0	$\|0\rangle$
$\hbar\omega$	$\|1\rangle$
$2\hbar\omega$	$\|2\rangle$
$\vdots$	$\vdots$
$E_n = n\hbar\omega$	$\|n\rangle$
$\vdots$	$\vdots$

$$\hat{H} = \sum_{n=0}^{\infty} E_n |n\rangle\langle n|$$

$$\hat{H} = \sum_{n=0}^{\infty} n\hbar\omega\, |n\rangle\langle n|$$

We can now apply the general framework to specific states and their measurements.

1 Gaussian states

Consider the following state for (monochromatic light)

$$|\Psi\rangle = A\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,.$$

where α is a constant.

a) Determine the normalization constant A.

b) Suppose that the photon number is measured. List the outcomes and their probabilities.

c) Suppose that the energy is measured. List the outcomes and their probabilities.

d) Determine the expectation value of the energy.

Answer: a) $\langle\Psi|\Psi\rangle = 1$

Then $\langle\Psi| = A^* \sum_{n=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} \langle n|$

Write $|\Psi\rangle$ with a different summation index:

$$|\Psi\rangle = A \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle$$

Then

$$\langle\Psi|\Psi\rangle = A^*A \sum_{m,n} \frac{\alpha^{*n}\alpha^m}{\sqrt{m!}\sqrt{n!}} \underbrace{\langle n|m\rangle}_{\delta_{mn}}$$

$$= |A|^2 \sum_n \frac{\alpha^{*n}\alpha^n}{n!}$$

$$= |A|^2 \sum_n \frac{(|\alpha|^2)^n}{n!}$$

$$= |A|^2 e^{|\alpha|^2}$$

Thus $\langle\Psi|\Psi\rangle = 1 \Rightarrow |A|^2 = e^{-|\alpha|^2}$

$$\Rightarrow A = e^{-|\alpha|^2/2}$$

Thus

$$|\Psi\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

b) The outcomes are $0, 1, 2, \ldots, n, \ldots$ Then

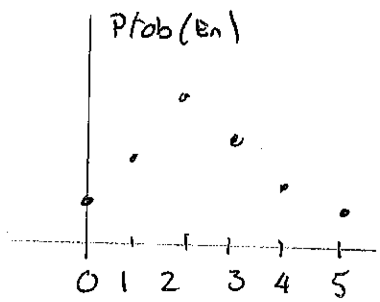
$$\text{Prob}(n) = |\langle n|\Psi\rangle|^2$$

$$= \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right|^2 = \frac{e^{-|\alpha|^2}}{n!} \alpha^{*n}\alpha^n$$

$$\Rightarrow \text{Prob}(n) = (|\alpha|^2)^n \frac{e^{-|\alpha|^2}}{n!}$$

c) The possible energies are $E_n = n\hbar\omega$ where $n = 0, 1, 2 \ldots$ By the above,

$$\text{Prob}(E_n) = \text{Prob}(n) = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}$$



d) $$\langle E\rangle = \sum E_n \text{Prob}(E_n)$$

$$= \sum_{n=0}^{\infty} n\hbar\omega \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}$$

$$= \hbar\omega \sum_{n=1}^{\infty} \frac{(|\alpha|^2)^n}{(n-1)!} e^{-|\alpha|^2} \qquad \text{since } \frac{n}{n!} = \frac{1}{(n-1)!}$$

$$= e^{-|\alpha|^2}\hbar\omega \sum_{m=0}^{\infty} \frac{(|\alpha|^2)^{m+1}}{m!} \qquad \text{let } m = n-1$$

$$= |\alpha|^2 e^{-|\alpha|^2}\hbar\omega \underbrace{\sum_{m=0}^{\infty} \frac{(|\alpha|^2)^m}{m!}}_{e^{|\alpha|^2}}$$

$$\Rightarrow \langle E\rangle = \hbar\omega|\alpha|^2$$

In general, for an infinite dimensional system

Consider general states:

$$|\Psi\rangle = \sum a_n |\phi_n\rangle$$

$$|\Phi\rangle = \sum b_n |\phi_n\rangle$$

Then

$$a_n = \langle \phi_n | \Psi \rangle$$

$$\langle \Phi | \Psi \rangle = \sum_n b_n^* a_n$$

Proof: First $\langle \phi_n | \Psi \rangle = \langle \phi_n | \sum_m a_m |\phi_m\rangle$

$$= \sum_m a_m \underbrace{\langle \phi_n | \phi_m \rangle}_{\delta_{nm}} = a_1 \delta_{n1} + a_2 \delta_{n2} + \dots$$

$$= a_n$$

Second

$$\langle \Phi | = \sum_m b_m^* \langle \phi_m |$$

$$\Rightarrow \langle \Phi | \Psi \rangle = \sum_{n,m} b_m^* a_n \underbrace{\langle \phi_m | \phi_n \rangle}_{\delta_{mn}}$$

$$= \sum_n b_n^* a_n \quad \blacksquare$$

Note that the latter sum is

$$\langle \Phi | \Psi \rangle = b_1^* a_1 + b_2 a_2^* + \dots$$

2 Particle in a infinite well

Consider a particle with mass m in an infinite one-dimensional well with width L. The possible outcomes of an energy measurement turn out to be

$$E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2$$

for $n = 1, 2, \ldots$. Let $|\phi_n\rangle$ be the state associated with energy outcome E_n. Consider the states

$$|\Psi_1\rangle := \frac{1}{\sqrt{2}}\left(|\phi_1\rangle + i\,|\phi_2\rangle\right)$$
$$|\Psi_2\rangle := \frac{1}{\sqrt{2}}\left(|\phi_1\rangle - i\,|\phi_2\rangle\right).$$

a) Verify that these are orthonormal.

b) Suppose that a system is in either of these states and the energy is measured. List the outcomes and the probabilities with which they occur.

c) Suppose that the particle is known to be in one of these states. Could one decide which it is with certainty by using energy measurements? Is there some measurement that exists which would allow one to decide the state with certainty?

d) This can be made more concrete by considering the operator

$$\hat{U} = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle\langle\phi_1| - i\,|\phi_1\rangle\langle\phi_2| + i\,|\phi_2\rangle\langle\phi_1| + \overset{0}{i}\,|\phi_2\rangle\langle\phi_2|\right) + \sum_{j>2}|\phi_j\rangle\langle\phi_j|,$$

which can be shown to be unitary. Suppose that prior to the evolution describe by this, the systems is either in state $|\Psi_1\rangle$ or $|\Psi_2\rangle$. Determine the state after the evolution. Use this to construct the measurement associated with the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$.

Answer: a)

$$\langle\Psi_1|\Psi_1\rangle = \frac{1}{\sqrt{2}}\left[\langle\phi_1| - i\langle\phi_2|\right]\frac{1}{\sqrt{2}}\left[|\phi_1\rangle + i|\phi_2\rangle\right]$$
$$= \frac{1}{2}\left[\overset{1}{\langle\phi_1|\phi_1\rangle} + i\overset{0}{\langle\phi_1|\phi_2\rangle} - i\overset{0}{\langle\phi_2|\phi_1\rangle} + \overset{1}{\langle\phi_2|\phi_2\rangle}\right] = 1$$

$$\langle\Psi_2|\Psi_2\rangle = \frac{1}{\sqrt{2}}\left[\langle\phi_1| + i\langle\phi_2|\right]\frac{1}{\sqrt{2}}\left[|\phi_1\rangle - i|\phi_2\rangle\right]$$
$$= \frac{1}{2}\left[\overset{1}{\langle\phi_1|\phi_1\rangle} + i\overset{0}{\langle\phi_2|\phi_1\rangle} - i\overset{0}{\langle\phi_1|\phi_2\rangle} + \overset{1}{\langle\phi_2|\phi_2\rangle}\right] = 1$$

$$\langle\Psi_1|\Psi_2\rangle = \frac{1}{2}\left[\langle\phi_1| - i\langle\phi_2|\right]\left[|\phi_1\rangle - i|\phi_2\rangle\right]$$
$$= \frac{1}{2}\left[\overset{1}{\langle\phi_1|\phi_1\rangle} - i\overset{0}{\langle\phi_2|\phi_1\rangle} - i\overset{0}{\langle\phi_1|\phi_2\rangle} - \overset{1}{\langle\phi_2|\phi_2\rangle}\right] = 0$$

They are orthonormal

b) The outcomes are $E_1, E_2, E_3, \ldots$ Then

$$\text{Prob}(E_j) = |\langle \phi_j | \Psi \rangle|^2$$

Clearly this is zero for $j = 3, 4, \ldots$ Then for $j = 1$

$$\text{Prob}(E_1) = |\langle \phi_1 | \Psi \rangle|^2$$

Now for $|\Psi_1\rangle$

$$\langle \phi_1 | \Psi_1 \rangle = \langle \phi_1 | \frac{1}{\sqrt{2}} \left[|\phi_1\rangle + i |\phi_2\rangle \right] = \frac{1}{\sqrt{2}} \left[\overset{1}{\cancel{\langle \phi_1 | \phi_1 \rangle}} + i \overset{0}{\cancel{\langle \phi_1 | \phi_2 \rangle}} \right]$$

$$= \frac{1}{\sqrt{2}}$$

So

$\text{Prob}(E_1) = \frac{1}{2}$

Then

$\text{Prob}(E_2) = \frac{1}{2}$

$\Big\}$ For state $|\Psi_1\rangle$ $\quad \text{Prob}(E_1) = \frac{1}{2}$, $\text{Prob}(E_2) = \frac{1}{2}$

For $|\Psi_2\rangle$

$$\langle \phi_1 | \Psi_2 \rangle = \frac{1}{\sqrt{2}} \Rightarrow \text{Prob}(E_1) = \frac{1}{2}$$

$\Big\}$ For state $|\Psi_2\rangle$ $\quad \text{Prob}(E_1) = \frac{1}{2}$, $\text{Prob}(E_2) = \frac{1}{2}$.

c) No the probabilities, and hence the statistics are identical

d) If initially in state. $|\Psi_1\rangle$ then after the evolution the state is

$$\hat{U} |\Psi_1\rangle = \frac{1}{\sqrt{2}} \left[|\phi_1\rangle\langle\phi_1| - i |\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1| + i |\phi_2\rangle\langle\phi_2| \right] \frac{1}{\sqrt{2}} \left[|\phi_1\rangle + i |\phi_2\rangle \right]$$

$$+ \cancel{\sum_{j>2} |\phi_j\rangle\langle\phi_j| \frac{1}{\sqrt{2}} \left[|\phi_1\rangle + i |\phi_2\rangle \right]} \; 0$$

$$= \frac{1}{2} \Big[|\phi_1\rangle \overset{1}{\cancel{\langle\phi_1|\phi_1\rangle}} + i |\phi_1\rangle \overset{0}{\cancel{\langle\phi_1|\phi_2\rangle}} - i |\phi_1\rangle \overset{0}{\cancel{\langle\phi_2|\phi_1\rangle}} + |\phi_1\rangle \overset{1}{\cancel{\langle\phi_2|\phi_2\rangle}}$$

$$|\phi_2\rangle \overset{1}{\cancel{\langle\phi_1|\phi_1\rangle}} + i |\phi_2\rangle \overset{0}{\cancel{\langle\phi_1|\phi_2\rangle}} + i |\phi_2\rangle \overset{0}{\cancel{\langle\phi_2|\phi_1\rangle}} - |\phi_2\rangle \overset{1}{\cancel{\langle\phi_2|\phi_2\rangle}} \Big]$$

$$= |\phi_1\rangle.$$

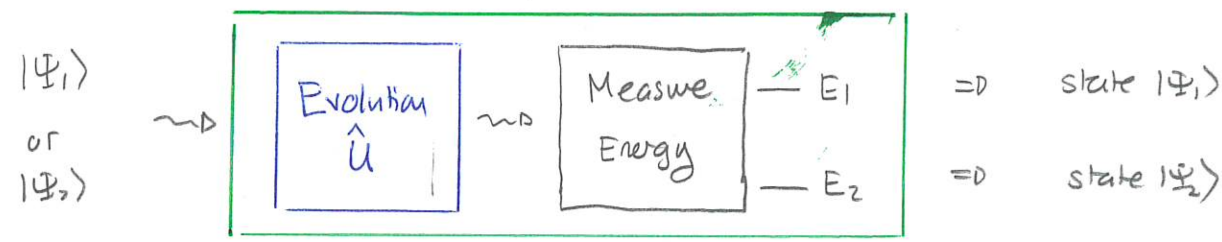
If $|\Psi_2\rangle$ is the initial state, then after evolution the state is

$$\hat{U}|\Psi_2\rangle = \frac{1}{\sqrt{2}}\left[|\phi_1\rangle\langle\phi_1| - i|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1| + i|\phi_2\rangle\langle\phi_2|\right]\frac{1}{\sqrt{2}}\left[|\phi_1\rangle - i|\phi_2\rangle\right]$$

$$= \frac{1}{2}\left[|\phi_1\rangle + (-i)^2|\phi_1\rangle + |\phi_2\rangle - (+i)^2|\phi_2\rangle\right]$$

$$= |\phi_2\rangle$$

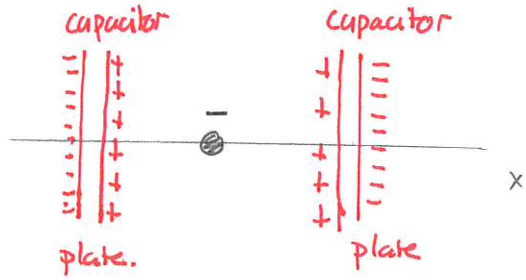
So we could do



Measurement!

Position and momentum measurements

We will need to consider particles which can move in one or more spatial dimension. A charged particle trapped in an infinite well could be an example. In this case we could



* measure the particle's energy
* measure the " position
* measure the " momentum

...

In order to answer questions about the energy we would need to construct the Hamiltonian. A further question would be

* how can one construct the Hamiltonian from the comparable classical energy

$$E = \frac{p^2}{2m} + V(x).$$

In order to answer these we will need to describe

* position states + measurement outcomes
* position observable
* momentum states + measurement outcomes
* momentum observable

The first complication here is that the measurement outcome possibilities are continuous. This means that we cannot make any statement about the probability of a particular outcome, e.g

$$\text{Prob}\,(x = 6.3) = ?$$

or $\text{Prob}\,(x = -12.6) = ?$

Rather we can describe the probability of an outcome in a given range. So we can get

what is probability that measurement gives outcome here?

$$\text{Prob}\begin{pmatrix}\text{position is in} \\ \text{range } x \to x+dx\end{pmatrix}$$

or

$$\text{Prob}\left(a \leq x \leq b\right)$$

The route to answering these questions is via a <u>probability density</u> $P(x)$. This is a quantity that roughly means

$$\text{Prob}\begin{pmatrix}\text{position outcome in} \\ \text{range } x \to x+dx\end{pmatrix} = P(x)\,dx$$

or more precisely

$$\boxed{\text{Prob}\left(a \leq x \leq b\right) = \int_a^b P(x)\,dx}$$

What quantum theory will do is.

Provide state of system $|\Psi\rangle$ → Construct wavefunction from state $\Psi(x)$ → Construct probability density for position measurements $P(x) = |\Psi(x)|^2$

Process takes state + produces prob density

→ Determine probabilities and statistics of measurement outcomes