Phys 321 2023

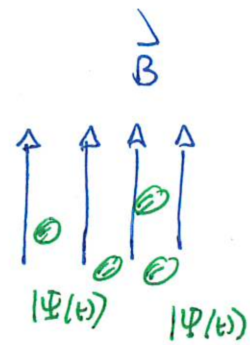
Lecture 13

Thurs: Seminar

Fri: HW 5pm

Tues: Read Section III pg 106 -
~~Section~~ Ch IV 1 pg 113 - 120

Evolution of expectation values

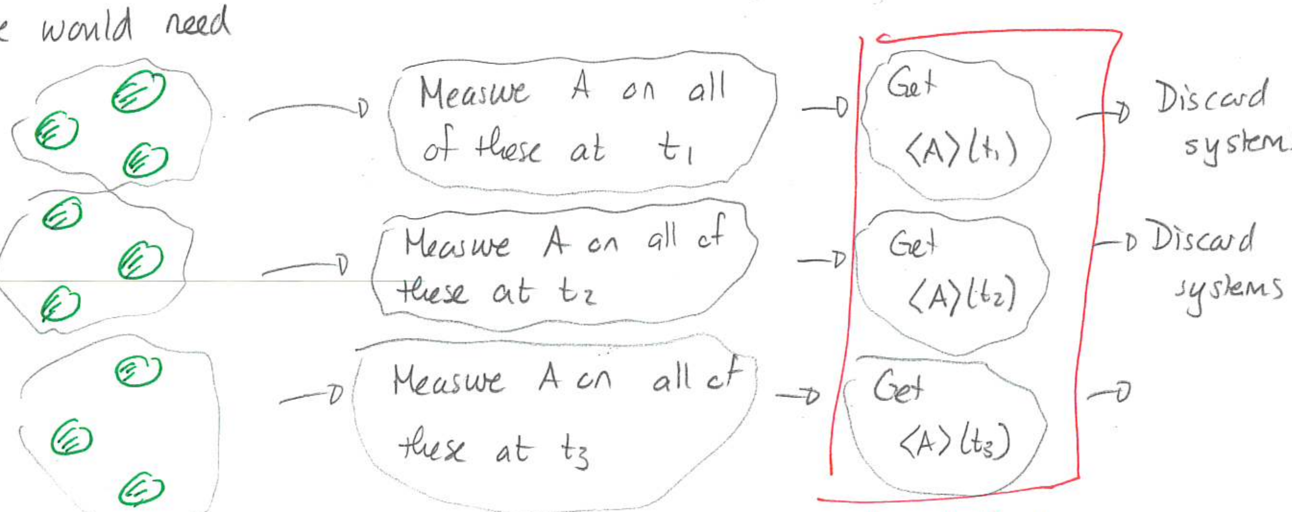


We can envisage a collection of spin-$\frac{1}{2}$ particles all subject to the same evolution. This can be arranged so that, at any time the state of each particle is the same as all others and is $|\Psi(t)\rangle$. A measurement could be performed on each particle at the same time and an expectation value could be determined. How do these expectation values depend on time?

Let A be the quantity to measure and $\hat{A}$ be the associated observable. Then the expectation value depends on time:

$$\langle A\rangle(t) = \langle \Psi(t)|\,\hat{A}\,|\Psi(t)\rangle$$

Note that if we wanted to perform such a measurement at various times we would need



Record of A over time

Entire ensemble in same initial state $(t=0)$ and evolve same way...

Then Ehrenfest's theorem predicts how this expectation value evolves:

$$\boxed{\frac{d}{dt}\langle A\rangle = \langle\Psi(t)|\frac{\partial\hat{A}}{\partial t}|\Psi(t)\rangle + \frac{i}{\hbar}\langle\Psi(t)|[\hat{H},\hat{A}]|\Psi(t)\rangle}$$

Proof: $\frac{d}{dt}\langle A\rangle = \frac{d}{dt}\left[\langle\Psi|\hat{A}|\Psi\rangle\right]$

$$= \frac{d\langle\Psi|}{dt}\hat{A}|\Psi\rangle + \langle\Psi|\frac{\partial\hat{A}}{\partial t}|\Psi\rangle + \langle\Psi|\hat{A}\frac{d|\Psi\rangle}{dt}.$$

Then $i\hbar\frac{d|\Psi\rangle}{dt} = \hat{H}|\Psi\rangle \Rightarrow \frac{d|\Psi\rangle}{dt} = \frac{-i}{\hbar}\hat{H}|\Psi\rangle.$

Also $$\frac{d\langle\Psi|}{dt} = \left(\frac{d|\Psi\rangle}{dt}\right)^{\dagger} = \left(\frac{-i}{\hbar}\hat{H}|\Psi\rangle\right)^{\dagger} = \frac{i}{\hbar}\underbrace{|\Psi\rangle^{\dagger}}_{\langle\Psi|}\underbrace{\hat{H}^{\dagger}}_{\hat{H}}$$

$$= \frac{i}{\hbar}\langle\Psi|\hat{H}$$

Thus:

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle\Psi|\hat{H}\hat{A}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|\hat{A}\hat{H}|\Psi\rangle + \langle\Psi|\frac{\partial\hat{A}}{\partial t}|\Psi\rangle$$

$$= \langle\Psi|\frac{\partial\hat{A}}{\partial t}|\Psi\rangle + \frac{i}{\hbar}\langle\Psi|[\hat{H},\hat{A}]|\Psi\rangle \qquad \boxtimes$$

1 3 **Time evolution of spin expectation values**

Consider an ensemble of spin-1/2 particles, each subject to the magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$. Denote the resulting Hamiltonian

$$\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}_x.$$

Determine equations for the time evolution of $\langle S_x \rangle$ and $\langle S_z \rangle$.

Answer: For $\langle S_x \rangle$.

$$\frac{d\langle S_x\rangle}{dt} = \frac{i}{\hbar}\langle\Psi|[\hat{H},\hat{S}_x]|\Psi\rangle = \frac{i\omega}{2}\langle\Psi|[\hat{\sigma}_x,\tfrac{\hbar}{2}\hat{\sigma}_x]|\Psi\rangle$$

$$= \frac{i\omega\hbar}{4}\langle\Psi|[\hat{\sigma}_x,\hat{\sigma}_x]|\Psi\rangle$$

But $[\hat{\sigma}_x,\hat{\sigma}_x] = \hat{\sigma}_x\hat{\sigma}_x - \hat{\sigma}_x\hat{\sigma}_x = 0$

$$\Rightarrow \frac{d\langle S_x\rangle}{dt} = 0 \quad \Rightarrow \quad \langle S_x\rangle = \text{constant}$$

For $\langle S_z \rangle$

$$\frac{d\langle S_z\rangle}{dt} = \frac{i}{\hbar}\langle\Psi|[\tfrac{\hbar\omega}{2}\hat{\sigma}_x,\tfrac{\hbar}{2}\hat{\sigma}_z]|\Psi\rangle = \frac{i\omega\hbar}{4}\langle\Psi|[\hat{\sigma}_x,\hat{\sigma}_z]|\Psi\rangle$$

But $[\hat{\sigma}_x,\hat{\sigma}_z] = \hat{\sigma}_x\hat{\sigma}_z - \hat{\sigma}_z\hat{\sigma}_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} - \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$$= \begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&1\\-1&0\end{pmatrix} = 2\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$= -2i\hat{\sigma}_y$$

$$\Rightarrow \frac{d\langle S_z\rangle}{dt} = \frac{i\omega\hbar}{4}(-2i)\langle\Psi|\hat{\sigma}_y|\Psi\rangle = \omega\,\underbrace{\langle\Psi|\tfrac{\hbar}{2}\sigma_y|\Psi\rangle}_{\langle S_y\rangle}$$

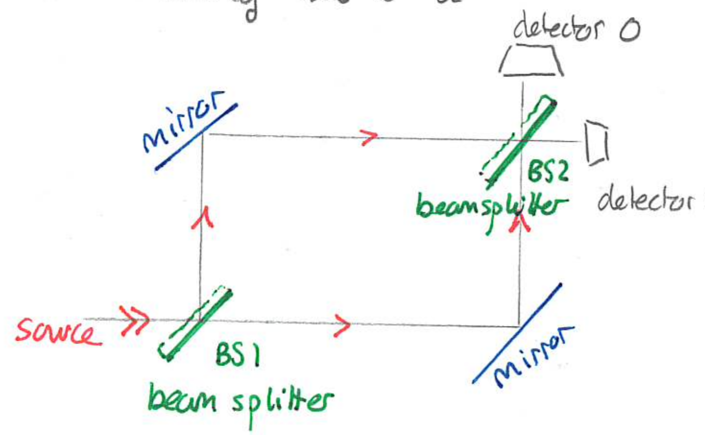
$$\Rightarrow \frac{d\langle S_z\rangle}{dt} = \omega\langle S_y\rangle$$

Photon Interference Experiments

The framework for describing the physics of spin-½ particles contains elements that are specific to spin-½ particles: e.g:

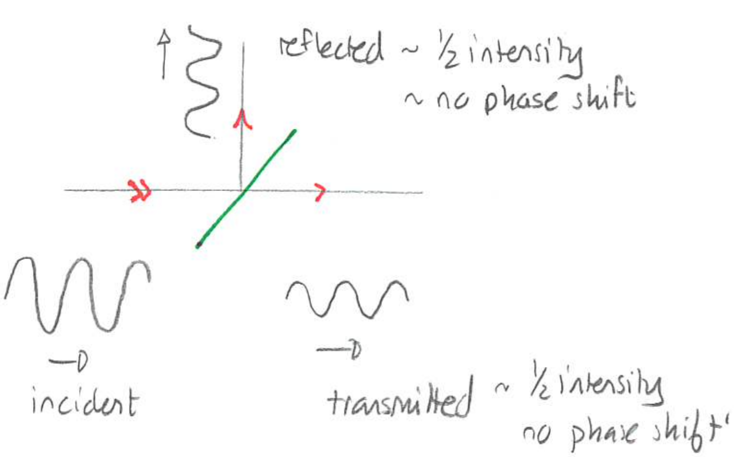
* measurement outcomes $S_n = \pm \hbar/2$
* representation of states $|\Psi\rangle = |+\hat{n}\rangle$ in terms of unit vector directions.
* commutation relations for observables
* unitary generation via magnetic fields / rotations.

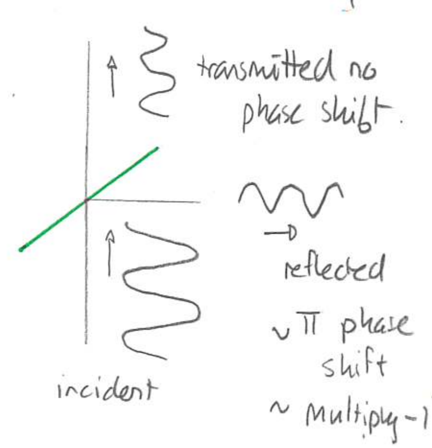
However, it contains general elements that apply to all quantum systems. We will illustrate these in a completely different setting: interference of light / photons. One set-up for demonstrating this is a Mach Zehnder interferometer which consists of beamsplitters an mirrors arranged as illustrated.



Demo: QuVis MZ

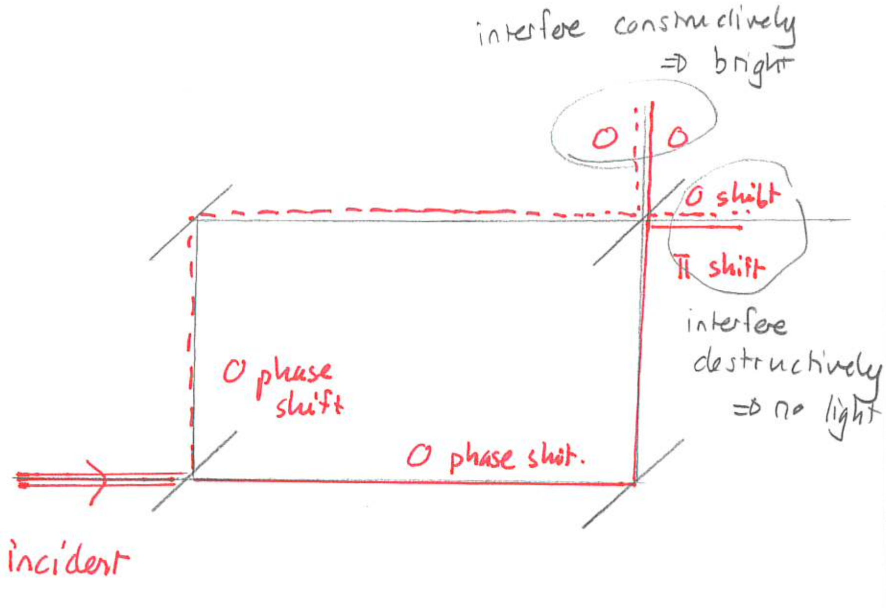
Light that is incident on a beam splitter is partly reflected and partly transmitted. There are various ways that light can travel from the source to either detector. In classical wave optics this can be analyzed in terms of superposition or inteference of waves. The key is the beamsplitter





We can track the waves through the two arms and we will find that all of the light arrives in one detector.

This experiment can be done with low-intensity light that must be described by single photons. A key rule in describing this is that



a single photon cannot be split at either beamsplitter.

We will need to use quantum theory, describing:

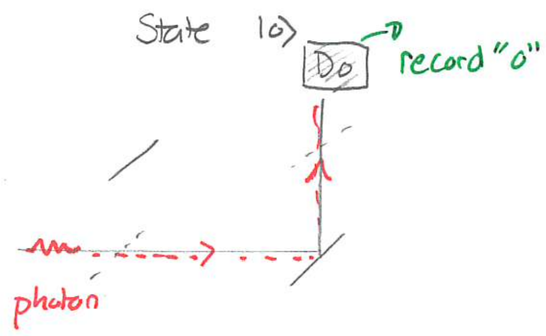
* states
* measurements
* evolution operators.

Here, with a single photon the measurement consists of two outcomes:

* photon arrives in D_0
* photon " in D_1

} Answers question: "Does photon arrive in D_0 or in D_1"

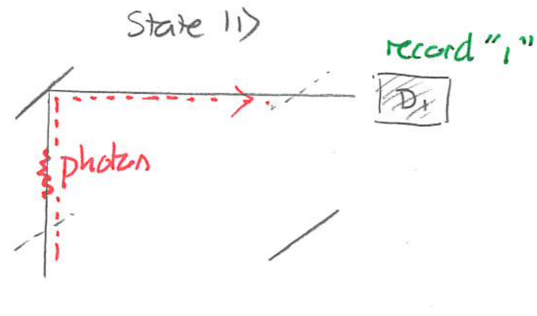
So we need two basic states that produce these outcomes with certainty. We can describe these when the beamsplitters are both absent



Photon traverses lower arm arrives in D_0 with certainty

$|0\rangle \rightsquigarrow \begin{pmatrix}1\\0\end{pmatrix}$

label



Photon traverses upper arm and arrives in D_1 with certainty

$|1\rangle \rightsquigarrow \begin{pmatrix}0\\1\end{pmatrix}$

label

Then: $\{|0\rangle, |1\rangle\}$ are orthonormal and constitute a basis for all possible states. So

$$\langle 0|0\rangle = 1 \qquad \langle 0|1\rangle = 0$$
$$\langle 1|1\rangle = 1 \qquad \langle 1|0\rangle = 0$$

The general state has form:

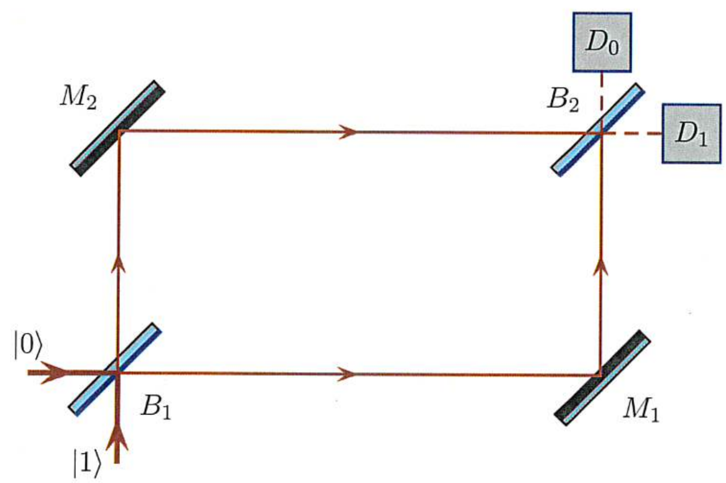
$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle$$

where

$$|c_0|^2 + |c_1|^2 = 1$$

2 Photon states in a Mach-Zehnder interferometer

A Mach-Zehnder interferometer consists of an arrangement of two beam splitters, B_1 and B_2, two mirrors, M_1 and M_2, and two detectors, D_0 and D_1, as illustrated below. Note that the reflective side of B_1 is down and right and that of B_2 is up and left. Ignore the thickness of the glass in the beam-splitters and assume that they reflect 50% of the beam and transmit 50%.



The orthonormal input states $\{|0\rangle, |1\rangle\}$ are indicated in the figure. The effects of B_1 and B_2 can be described by the operators

$$\hat{U}_{B_1} := \frac{1}{\sqrt{2}}\Big\{|0\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\Big\} \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\hat{U}_{B_2} := \frac{1}{\sqrt{2}}\Big\{|0\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\Big\} \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

where the matrices indicate representations in the basis $\{|0\rangle, |1\rangle\}$. The following questions all assume that a single photon initially in the state $|0\rangle$ is incident upon the apparatus.

a) Suppose that neither beam splitter is in place. Determine the probabilities with which the photon will arrive at each detector.

b) Suppose that only B_1 is in place. Determine the probabilities with which the photon will arrive at each detector.

c) Suppose that both beam splitters are in place. Determine the state of the photon immediately after B_2 and the probabilities with which it will arrive at each detector.

Answer: a) Then the state is $|0\rangle$. We can determine probabilities via the standard rule:

Detector	Associated State	Probability
0	$\|0\rangle$	$\|\langle 0\|\Psi\rangle\|^2 = 1$
1	$\|1\rangle$	$\|\langle 1\|\Psi\rangle\|^2 = 0$

Then $\langle 0|\Psi\rangle = \langle 0|0\rangle = 1$, $\langle 1|\Psi\rangle = \langle 1|0\rangle = 0$ $\Rightarrow$ Arrives at 0 with certainty.

b) After B_1

$$|\Psi_1\rangle = U_{B_1}|0\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

Then:

$$\text{Prob}(0) = |\langle 0|\Psi_1\rangle|^2$$

and $\langle 0|\Psi_1\rangle = (1\ \ 0)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}$. Thus

$$\text{Prob}(0) = \frac{1}{2}$$

Similarly $\text{Prob}(1) = \frac{1}{2}$

$$\text{Prob}(0) = \frac{1}{2}$$
$$\text{Prob}(1) = \frac{1}{2}$$

c) After B_1 $|\Psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

Then after B_2 state is $|\Psi_2\rangle = \hat{U}_{B_2}|\Psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$
$= |0\rangle$.

Thus $\text{Prob}(0) = 1$

$\text{Prob}(1) = 0$

as before.

General Quantum Theory Framework

We have the following basis framework

System

Physical system with specified characteristics

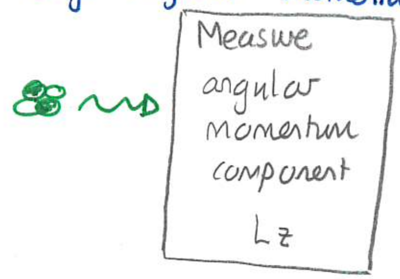
e.g. Helium atom

Described by state: $|\Psi\rangle$

Measurements

Specification of measurement that can be performed on the system

e.g. Angular momentum



List outcomes and states:

L_1	$\lvert\phi_1\rangle$
L_2	$\lvert\phi_2\rangle$
$\vdots$	$\vdots$

Evolution

Describes how state transforms as a result of interactions

laser light incident on Helium

$$|\Psi_{initial}\rangle \rightsquigarrow |\Psi_{final}\rangle = \hat{U}\,|\Psi_{initial}\rangle$$

evolution operator

The same general framework applies to all quantum systems and we will provide the basic ingredients.

Technical note: This framework is valid for pure states and unitary evolutions. There is an extension to general noisy (mixed) states and noisy non-unitary evolutions (quantum operations).

1) States of a system

The state of the system is described by a ket $|\Psi\rangle$. This is an element of a complex vector space:

$$|\Psi\rangle \rightsquigarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

$c_1, c_2 \ldots$ complex numbers.

The dimension of the vector space fixes certain properties of the system such as the maximum number of distinct, mutually incompatible outcomes of any single measurement (a maximal measurement)

The vector space has an inner product and this can be computed by conversion from a ket to a bra

$$|\Phi\rangle \rightsquigarrow \begin{pmatrix} d_1 \\ d_2 \\ \vdots \end{pmatrix} \Rightarrow \langle\Phi| = |\Phi\rangle^\dagger \rightsquigarrow \left(d_1^* \; d_2^* \; \ldots\ldots\right)$$

All states are normalized: $\langle\Psi|\Psi\rangle = 1$

2) Measurements

A measurement is described by

* a set of outcomes
* a state associated with each outcome

Measurement "A"

outcome	associated state
a_1	$\|a_1\rangle$
a_2	$\|a_2\rangle$
a_3	$\|a_3\rangle$ ↳ label
$\vdots$	$\vdots$
real numbers	complex vectors

Here: "associated state" means

If a system is in state $|a_j\rangle$ then a measurement of A will yield a_j with certainty.

Then:

For a maximal measurement, the associated states form an orthonormal basis for the ket vector space.

A technical way to describe this basis requirement is the completeness relation:

$$\sum_{\text{all outcomes } j} |a_j\rangle\langle a_j| = \hat{I}$$

Then: the key measurement statement is:

If a system is in state $|\Psi\rangle$ then measurement of A will yield outcome a_j with probability

$$\text{Prob}(a_j) = |\langle a_j|\Psi\rangle|^2$$

state associated with a_j outcome

The observable associated with this measurement is.

$$\hat{A} = \sum_{\text{all } j} a_j |a_j\rangle\langle a_j|$$

This observable is Hermitian:

$$\hat{A}^\dagger = \hat{A}$$

Given any observable:

The outcomes and associated states satisfy the eigenvalue equation

$$\hat{A}|a_j\rangle = a_j|a_j\rangle$$

outcome

associated state

3) evolution

Any evolution of the system is described via a unitary linear operation:

If the initial state of a system is $|\Psi_i\rangle$ then the final state after evolution is

$$|\Psi_f\rangle = \hat{U}|\Psi_i\rangle$$

where $\hat{U}$ is a unitary operator:

$$\hat{U}^\dagger \hat{U} = \hat{I}$$

This depends on the details of the interaction.

When the state of a system depends on time then:

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

where $\hat{H}$ is the Hamiltonian (energy observable)

When the Hamiltonian is time-independent then

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar}$$

Similarly

If $|\phi_j\rangle$ is an energy eigenstate with energy E_j and

$$|\Psi_{initial}\rangle = \sum c_j |\phi_j\rangle$$

then

$$|\Psi(t)\rangle = \sum e^{-iE_jt/\hbar} c_j |\phi_j\rangle$$