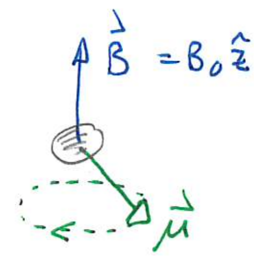
Fri: HW 5pm

Tues: Read My notes 4.7, 4.8

Time evolution of classical magnetic dipoles

A classical magnetic dipole in a magnetic field satisfies

$$\frac{d\vec{\mu}}{dt} = \frac{gQ}{2M}\,\vec{\mu}\times\vec{B}$$



We can arrange co-ordinates so that $\vec{B} = B_0\hat{z}$. Then the resulting differential equations have solution

$$\mu_x(t) = \cos(\omega t)\,\mu_x(0) + \sin(\omega t)\,\mu_y(0)$$

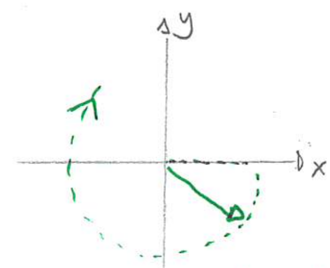
$$\mu_y(t) = -\sin\omega t\,\mu_x(0) + \cos(\omega t)\,\mu_y(0)$$

$$\mu_z(t) = \mu_z(0)$$

where $\omega = \frac{gQB}{2M}$. This describes a dipole that

* rotates around $\vec{B}$ with angular velocity ω
* this is called precession.

transverse component



Demo: Gyroscope bicycle wheel.

This can be expressed as a linear transformation that maps the initial state (dipole moment at $t=0$) to a later state (dipole moment at t)

$$\underbrace{\begin{pmatrix} \mu_x(t) \\ \mu_y(t) \\ \mu_z(t) \end{pmatrix}}_{\substack{\text{later state} \\ \vec{\mu}(t)}} = \underbrace{\begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{\text{operator } \hat{R}(t) \\ \text{maps earlier to later} \\ \text{state}}} \underbrace{\begin{pmatrix} \mu_x(0) \\ \mu_y(0) \\ \mu_z(0) \end{pmatrix}}_{\text{earlier state } \vec{\mu}(0)}$$

$$\hookrightarrow \quad \vec{\mu}(t) = \hat{R}(t)\,\vec{\mu}(0).$$

This operation is linear. That means

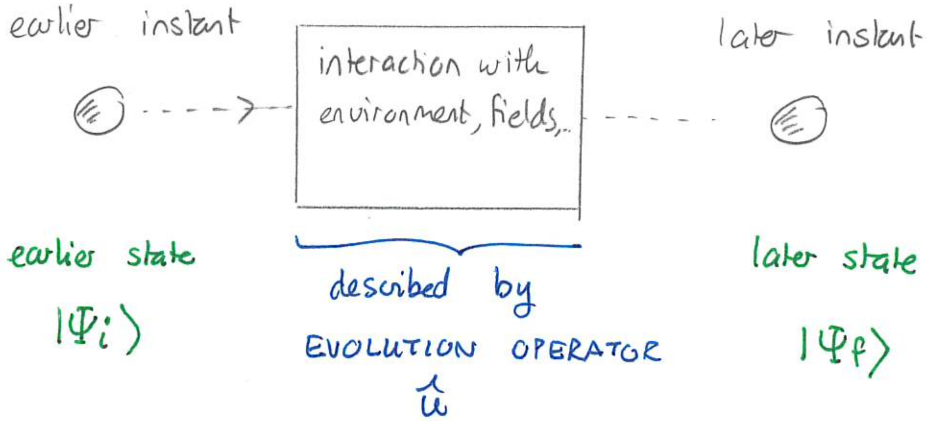
$$\vec{\mu}_1(0) + \vec{\mu}_2(0) \longrightarrow \vec{\mu}_1(t) + \vec{\mu}_2(t)$$

where $\vec{\mu}_i(t) = \hat{R}\,\vec{\mu}_i(0)$. We can view it abstractly as:

$$\underbrace{\text{particle at time } t=0}_{\text{earlier state } \vec{\mu}(0)} \qquad \underbrace{}_{\text{operation } R} \qquad \underbrace{\text{particle at time } t}_{\text{later state } \vec{\mu}(t) = R\vec{\mu}(0)}$$

Evolution of quantum systems

We can consider an analogous process for a quantum system:



We need a mathematical relationship between the later state and the earlier state that incorporates the effects of the interaction. An (axiomatic) requirement of quantum theory is that the relationship is linear. Thus

> For a given interaction, there exists an operator $\hat{U}$, depending on the details of the interaction so that
>
> $$|\Psi_f\rangle = \hat{U}\,|\Psi_i\rangle$$

The operator $\hat{U}$ is called the evolution operator and

1) it depends on the interaction
2) it does not depend on the initial / final states.

We now need:

1) to describe general requirements that <u>any</u> evolution operator satisfies
2) give examples of evolution operators
3) give ways to construct evolution operators based on descriptions of interactions with the environment.

Unitary operators

Any evolution operator must map a legitimate physical state to another legitimate physical state. The primary requirement for a legitimate state is that it is normalized. Thus

$$\langle \Psi_i | \Psi_i \rangle = 1 \qquad \text{and} \qquad \langle \Psi_f | \Psi_f \rangle = 1.$$

Then consider $\langle \Psi_f | = | \Psi_f \rangle^{+}$

$$= \left(\hat{U} | \Psi_i \rangle \right)^{+} = | \Psi_i \rangle^{+} \hat{U}^{+} = \langle \Psi_i | \hat{U}^{+}$$

So for all $|\Psi_i\rangle$

$$\underbrace{\langle \Psi_i | \hat{U}^{+}}_{\langle \Psi_f |} \underbrace{\hat{U} | \Psi_i \rangle}_{| \Psi_f \rangle} = 1$$

This will clearly be satisfied if $\hat{U}^{+}\hat{U} = \hat{I}$ since

$$\langle \Psi_i | \hat{U}^{+} \hat{U} | \Psi_i \rangle = \langle \Psi_i | \hat{I} | \Psi_i \rangle = \langle \Psi_i | \Psi_i \rangle = 1$$

A variety of linear algebra theorems then state that it can only be satisfied for all $|\Psi_i\rangle$ if $\hat{U}^{+}\hat{U} = \hat{I}$. Thus

$$\boxed{\text{Any evolution operator satisfies } \hat{U}^{+}\hat{U} = \hat{I}}$$

An operator for which $\hat{U}^{+}\hat{U} = \hat{I}$ is called <u>unitary</u>. Thus

$$\boxed{\text{Any evolution operator is unitary.}}$$

A converse assumption is

$$\boxed{\text{Any unitary operator corresponds to an evolution operator.}}$$

1 Characterization of evolution operators

Consider two proposed candidates for evolution operators:

$$\hat{U} = |+\hat{x}\rangle\langle+\hat{z}| - |-\hat{x}\rangle\langle-\hat{z}| \quad \text{and}$$
$$\hat{V} = |+\hat{x}\rangle\langle+\hat{z}| - |-\hat{z}\rangle\langle-\hat{z}|$$

a) Determine the matrix for each and verify which of these represents a legitimate evolution operator.

b) Suppose that the legitimate evolution operator acts on the initial states:

$$|+\hat{z}\rangle \qquad |-\hat{z}\rangle \qquad |+\hat{x}\rangle \qquad |-\hat{z}\rangle \quad |\hat{x}\rangle$$

Determine the ket that corresponds to the final state for each of these input states.

c) Describe the evolution geometrically in terms of its actions on these initial states. To do this, consider the unit vector labels in the various initial and final states.

Answer: a)

$$|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|-\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{1}{\sqrt{2}}|-\hat{z}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Then $\hat{U} \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\0 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix} \rightsquigarrow \hat{U}$$

Then $\hat{V} \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\1 & -\sqrt{2}\end{pmatrix} \rightsquigarrow \hat{V}$$

Now check for unitary: $\hat{U}^\dagger\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \hat{I}$

$$\hat{V}^\dagger\hat{V} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\0 & -\sqrt{2}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\1 & -\sqrt{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & -\sqrt{2}\\-\sqrt{2} & 2\end{pmatrix} = \begin{pmatrix}1 & -1/\sqrt{2}\\-\frac{1}{\sqrt{2}} & 1\end{pmatrix} \neq \hat{I}$$

So only $\hat{U}$ is legitimate.

b) $\hat{U}|+\hat{z}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\hat{x}\rangle \Rightarrow \boxed{\hat{U}|+\hat{z}\rangle = |+\hat{x}\rangle}$

$\hat{U}|-\hat{z}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -|-\hat{x}\rangle$

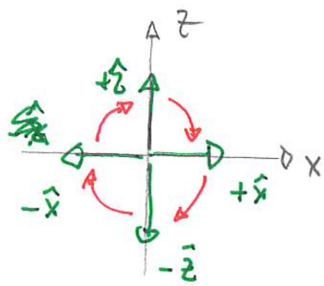
$\Rightarrow \boxed{\hat{U}|-\hat{z}\rangle = -|-\hat{x}\rangle}$

irrelevant global phase

$\hat{U}|+\hat{x}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-\hat{z}\rangle \Rightarrow \boxed{\hat{U}|+\hat{x}\rangle = |-\hat{z}\rangle}$

$\hat{U}|-\hat{x}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\hat{z}\rangle \Rightarrow \boxed{\hat{U}|+\hat{x}\rangle = |+\hat{z}\rangle}$

c) Consider the unitvector labels. The evolution appears to rotate the labels through $\pi/2$ about the $\hat{y}$-axis



We can think of the evolution geometrically as a rotation

There is an alternative strategy for checking that an operator is unitary and this does not require matrices. For example

$$\hat{U} = |+\hat{x}\rangle\langle+\hat{z}| - |-\hat{x}\rangle\langle-\hat{z}|$$

$$\Rightarrow \quad \hat{U}^\dagger = \left[|+\hat{x}\rangle\langle+\hat{z}| - |-\hat{x}\rangle\langle-\hat{z}|\right]^\dagger$$

$$= \left[|+\hat{x}\rangle\langle+\hat{z}|\right]^\dagger - \left[|-\hat{x}\rangle\langle-\hat{z}|\right]^\dagger$$

$$= \langle+\hat{z}|^\dagger \, |+\hat{x}\rangle^\dagger - \langle-\hat{z}|^\dagger \, |-\hat{x}^\dagger\rangle$$

$$= |+\hat{z}\rangle\langle+\hat{x}| - |-\hat{z}\rangle\langle-\hat{x}|$$

Now

$$\hat{U}^\dagger\hat{U} = \left[|+\hat{z}\rangle\langle+\hat{x}| - |-\hat{z}\rangle\langle-\hat{x}|\right]\left[|+\hat{x}\rangle\langle+\hat{z}| - |-\hat{x}\rangle\langle-\hat{z}|\right]$$

$$= |+\hat{z}\rangle\cancelto{1}{\langle+\hat{x}|+\hat{x}\rangle}\langle+\hat{z}| - |+\hat{z}\rangle\cancelto{0}{\langle+\hat{x}|-\hat{x}\rangle}\langle-\hat{z}| - |-\hat{z}\rangle\cancelto{0}{\langle-\hat{x}|+\hat{x}\rangle}\langle+\hat{z}|$$
$$+ |-\hat{z}\rangle\cancelto{1}{\langle-\hat{x}|-\hat{x}\rangle}\langle-\hat{z}|$$

$$= |+\hat{z}\rangle\langle+\hat{z}| + |-\hat{z}\rangle\langle-\hat{z}| \quad = \hat{I}$$

Then the following theorem guarantees this:

> If $\{|+\hat{n}\rangle, |-\hat{n}\rangle\}$ is an orthonormal basis for kets then
>
> $$|+\hat{n}\rangle\langle+\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}| = \hat{I}$$

This is called the <u>completeness relation</u>

Geometric interpretation for unitary operations on spin ½ particles

We have seen that for the evolution operator

$$\hat{U} = |+\hat{x}\rangle\langle+\hat{z}| - |-\hat{x}\rangle\langle-\hat{z}|$$

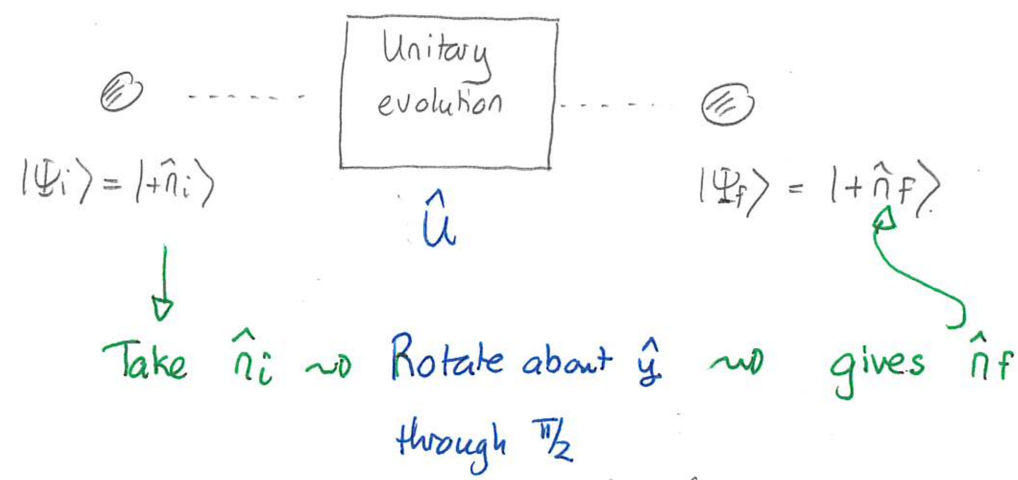
that the ket labels rotate. More specifically for any initial state there is $\hat{n}_i$ such that

$$|\Psi_i\rangle = |+\hat{n}_i\rangle$$

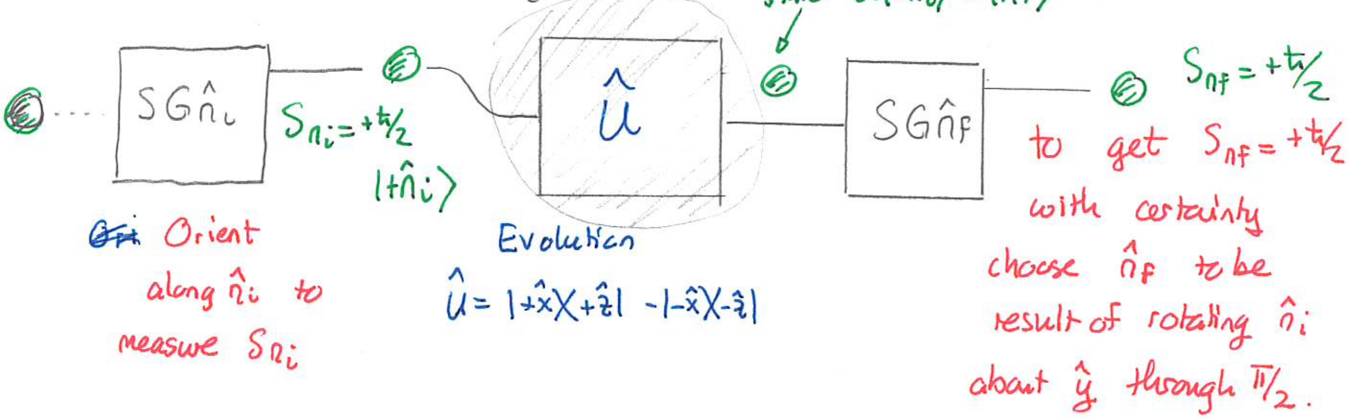
and similarly for any final state there is (ignoring a global phase) $\hat{n}_f$ such that

$$|\Psi_f\rangle = |+\hat{n}_f\rangle$$

Then schematically



In this sense the unitary operator in the example represents a rotation. In more physical terms (via measurements)



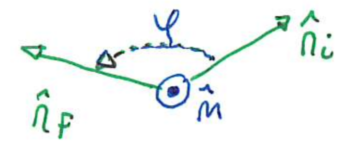
We can describe such processes via rotation operators on kets. This is structured as follows

Describe rotation

Specify axis of rotation $\hat{m}$ →

Specify angle of rotation $\hat{\varphi}$ →

Ordinary vector rotates c.c.w about $\hat{m}$ through angle φ



Then the comparable operation on kets is denoted

$$\hat{R}(\varphi\hat{m}) \equiv \text{rotation through } \varphi \text{ about } \hat{m}$$

actual operator: $\hat{R}$

describes rotation: $(\varphi\hat{m})$

Then this must map $|+\hat{n}_i\rangle \rightarrow |+\hat{n}_f\rangle$ where $\hat{n}_f$ is attained by rotating $+\hat{n}_i$ about $\hat{m}$ through angle φ. A general theorem then tells us that:

> The operation $\hat{R}(\varphi\hat{m})$ corresponding to a rotation about $\hat{m}$ through angle φ is
>
> $$\hat{R}(\varphi\hat{m}) = e^{-i\varphi/2}|+\hat{m}\rangle\langle+\hat{m}| + e^{i\varphi/2}|-\hat{m}\rangle\langle-\hat{m}|$$

Then $\hat{R}(\varphi\hat{m})|+\hat{n}_i\rangle = \text{global phase } |+\hat{n}_f\rangle$ where $\hat{n}_f$ is rotated about $\hat{m}$ through angle φ.

2 Rotation operator

Determine the matrix representing $\hat{R}(\frac{\pi}{2}\hat{\mathbf{y}})$ and compare it to

$$\hat{U} = |+\hat{\boldsymbol{x}}\rangle\langle+\hat{\boldsymbol{z}}| - |-\hat{\boldsymbol{x}}\rangle\langle-\hat{\boldsymbol{z}}|.$$

Answer:

$$R\left(\tfrac{\pi}{2}\hat{y}\right) = e^{-i\pi/4}|+\hat{y}\rangle\langle+\hat{y}| + e^{i\pi/4}|-\hat{y}\rangle\langle-\hat{y}|$$

$$= e^{-i\pi/4}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\frac{1}{\sqrt{2}}(1\ -i) + e^{i\pi/4}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\frac{1}{\sqrt{2}}(1\ i)$$

$$= e^{-i\pi/4}\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} + e^{i\pi/4}\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}(e^{-i\pi/4}+e^{i\pi/4}) & i(e^{i\pi/4}-e^{-i\pi/4})\\ i\underbrace{(e^{-i\pi/4}-e^{i\pi/4})}_{\cancel{-2is}\ -2i\sin\pi/4} & \underbrace{(e^{-i\pi/4}+e^{i\pi/4})}_{2\cos\pi/4}\end{pmatrix}$$

$$= \begin{pmatrix}\cos\pi/4 & i(i\sin\pi/4)\\ i(-i\sin\pi/4) & \cos\pi/4\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

This is the same as $\hat{u} = |+\hat{x}\rangle\langle+\hat{z}| - |-\hat{x}\rangle\langle-\hat{z}|$.

This is an example of a general rule for 2x2 unitary operators. One can prove.

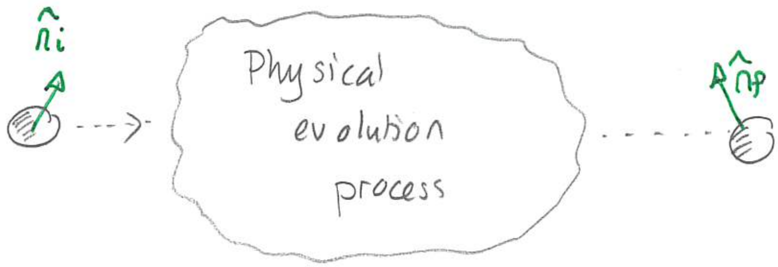
Let $\hat{U}$ be any unitary operator for a spin-½ system. Then there is a rotation with axis $\hat{m}$, angle φ and a real parameter α such that

$$\boxed{\hat{U} = e^{i\alpha}\hat{R}(\varphi\hat{m})}$$

The $e^{i\alpha}$ term provides an irrelevant global phase. Thus

Any unitary evolution of a spin-½ system is equivalent to a rotation

So



Initial state

$|\Psi_i\rangle = |+\hat{n}_i\rangle$

Unitary evolution

$\hat{U}$

Final state

$|\Psi_f\rangle = |+\hat{n}_f\rangle$

same as rotation

$\hat{R}(\varphi\hat{m})$ . . . . . . $|\Psi_f\rangle = \hat{R}(\varphi\hat{m})|+\hat{n}_i\rangle$