Phys 131 2023

Lecture 9

Tues: HW by 5pm

Thurs: Read ~~text.~~

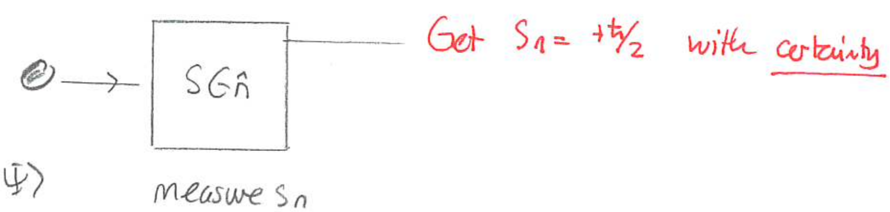
My notes: - later... 4.2 $\to$ 4.3

Compatible and incompatible measurements

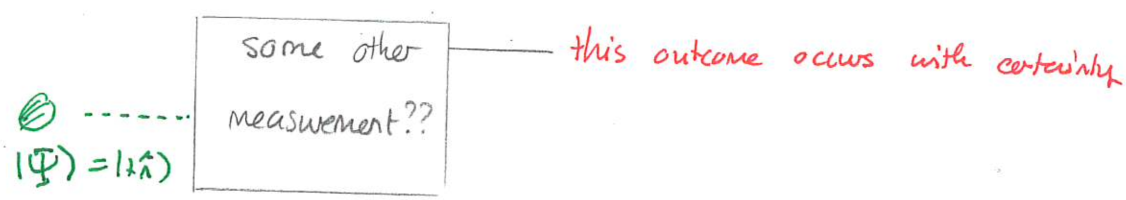
Consider a spin-$\frac{1}{2}$ particle in state $|\Psi\rangle$. Then there exists a direction $\hat{n}$ such that

$$|\Psi\rangle = |+\hat{n}\rangle$$

and this state has the interpretation:



We can ask whether there is any other measurement that yields one outcome with certainty for a particle in the same state $|\Psi\rangle$. So



If such a pair of measurements exists then we say that they are compatible.

The requirements for compatiblility can be stated in terms of the observable operators associated with the measurements. Recall that an observable $\hat{A}$ can be described by measurement outcomes via:

Solve eigenvalue equation.

$$\hat{A}|\phi\rangle = a|\phi\rangle \longrightarrow$$

Outcomes	States	Meas. Operators
a_1	$\vert\phi_1\rangle$	$\hat{P}_1 = \vert\phi_1\rangle\langle\phi_1\vert$
a_2	$\vert\phi_2\rangle$	$\hat{P}_2 = \vert\phi_2\rangle\langle\phi_2\vert$
a_3	$\vert\phi_3\rangle$	$\hat{P}_3 = \vert\phi_3\rangle\langle\phi_3\vert$ ..
$\vdots$		

For a given state $|\Psi\rangle$ we can get the probabilities

Then the expectation value is

Outcome	Probability
a_1	$p_1 = \langle\Psi\vert\hat{P}_1\vert\Psi\rangle$
a_2	$p_2 = \langle\Psi\vert\hat{P}_2\vert\Psi\rangle$
a_3	$p_3 = \langle\Psi\vert\hat{P}_3\vert\Psi\rangle$
$\vdots$	$\vdots$

$$\langle A\rangle = \sum_i a_i p_i$$

$$= \langle\Psi|\hat{A}|\Psi\rangle.$$

The uncertainty in the measurement will be

$$\Delta A = \sqrt{\sum p_i(a_i - \langle A\rangle)^2}$$

and it can be shown that

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$$

where

$$\langle A^2\rangle = \sum p_i a_i^2 \Rightarrow \langle A^2\rangle = \langle\Psi|\hat{A}^2|\Psi\rangle$$

If the state $|\Psi\rangle$ is such that only one outcome of the measurement of $\hat{A}$ occurs with certainty then $\Delta A = 0$.

1 Measurements on an ensemble in state $|+\hat{x}\rangle$

A spin-1/2 particle is in the state $|+\hat{x}\rangle$. Recall that

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and}$$
$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Suppose that S_z is measured. Determine $\langle S_z\rangle$ and ΔS_z.

b) Suppose that S_x is measured. Determine $\langle S_x\rangle$ and ΔS_x.

Answer: a) $\langle S_z\rangle = \langle\Psi|\hat{S}_z|\Psi\rangle$ $\qquad |\Psi\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{\hbar}{2}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{\hbar}{2} = 0$$

Then $\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{\langle S_z^2\rangle}$

Now

$$\langle S_z^2\rangle = \langle\Psi|\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}|\Psi\rangle$$

$$= \frac{\hbar^2}{4}\langle\Psi|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}|\Psi\rangle$$

$$= \frac{\hbar^2}{4}\langle\Psi|\Psi\rangle = \frac{\hbar^2}{4}.$$

$$\Rightarrow \quad \Delta S_z = \sqrt{\frac{\hbar^2}{4}} \quad \Rightarrow \quad \Delta S_z = \frac{\hbar}{2}$$

b) $\langle S_x \rangle = \langle +\hat{x} | \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} | +\hat{x} \rangle$

$$= \frac{1}{\sqrt{2}}(1\ 1)\ \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar}{4}(1\ 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{4}2$$

$$\Rightarrow \langle S_x \rangle = \frac{\hbar}{2}$$

$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2}$$

Now

$$\langle S_x^2 \rangle = \langle +\hat{x} | \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} | +\hat{x} \rangle = \frac{\hbar^2}{4}\langle +\hat{x} | \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} | +\hat{x} \rangle$$

$$= \frac{\hbar^2}{4}\langle +\hat{x} | +\hat{x} \rangle = \frac{\hbar^2}{4}.$$

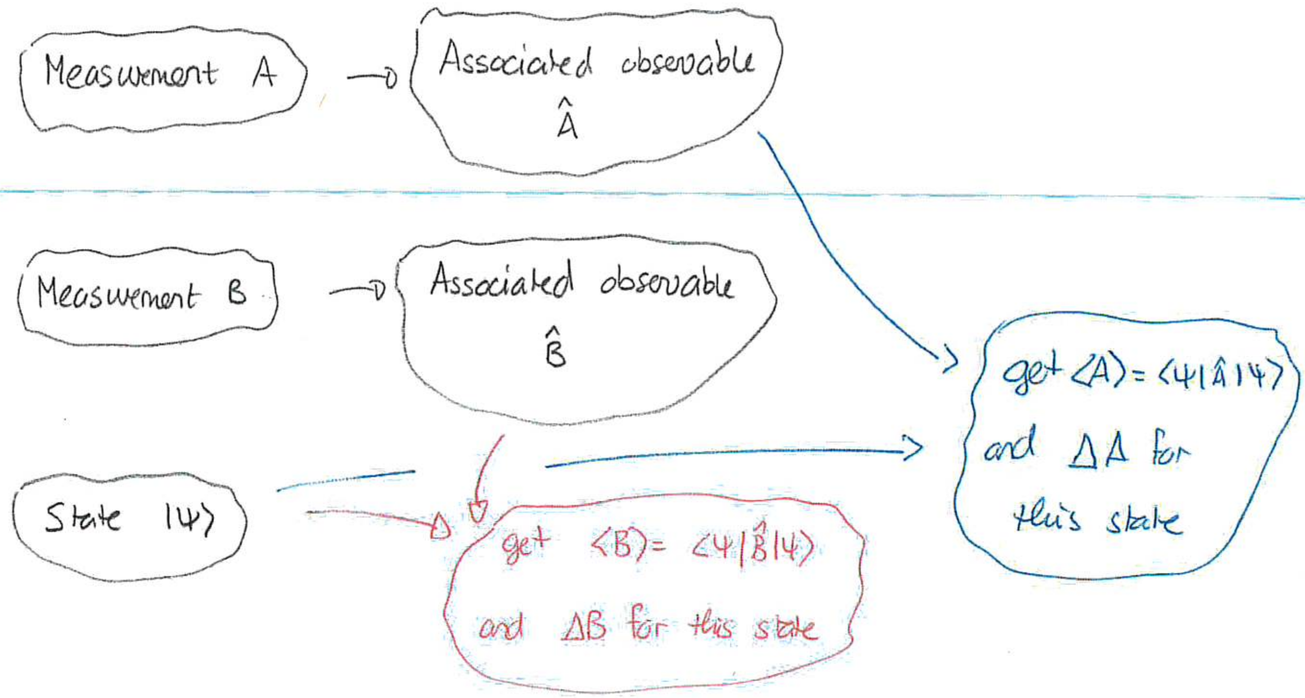
So

$$\Delta S_x = \sqrt{\frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2} = 0 \quad \boxtimes$$

Thus, for the state $|+\hat{x}\rangle$ a measurement of S_x yields one outcome $(+\frac{\hbar}{2})$ with certainty. However, a measurement of S_z does not yield one outcome with certainty.

So S_x and S_z are not compatible, for this state. Is there any state for which they are compatible?

The following yields a way to assess this. Consider two measurements, A and B.



Then a theorem states that

$$\boxed{(\Delta A)(\Delta B) \geqslant \frac{1}{2}\left|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right|}$$

where the commutator of the two observables is:

$$\boxed{[\hat{A},\hat{B}] := \hat{A}\hat{B} - \hat{B}\hat{A}}$$

We can compute this for various cases. Clearly if $[\hat{A},\hat{B}] \neq 0$ then there is some chance that no state will yield $\Delta A = 0$ and $\Delta B = 0$

The inequality is a statement of the Heisenberg uncertainty principle

Rather try this on the $|+\hat{y}\rangle$ state. Then

$\Delta S_z = \hbar/2$

$\Delta S_x = +\hbar/2$ $\qquad \frac{1}{2}|\langle [\,.. \,]\rangle| = \frac{\hbar^2}{2}\frac{1}{2}$.

2 Uncertainty principle for spin-1/2

a) Determine the commutator, $[\hat{S}_z, \hat{S}_x]$.

b) Verify that the Heisenberg uncertainty principle is valid for measurements of S_x and S_z on $|+\hat{x}\rangle$.

Answer: a) $[\hat{S}_z, \hat{S}_x] = \hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z$

$$\hat{S}_z\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$\hat{S}_x\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$[\hat{S}_z, \hat{S}_x] = \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

Note $[\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$

b) We had for $|\Psi\rangle = |+\hat{x}\rangle$ that

$\Delta S_x = 0$

$\Delta S_z = \frac{\hbar}{2}$

So

$\Delta S_x \Delta S_z = 0$

$$\langle +\hat{x}|[\hat{S}_z, \hat{S}_x]|+\hat{x}\rangle = \frac{1}{\sqrt{2}}(1\ 1)\frac{\hbar^2}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{\hbar^2}{4}(1\ 1)\begin{pmatrix}1\\ -1\end{pmatrix} = 0$$

Thus $\Delta S_x \Delta S_z \geq \frac{1}{2}|\langle +\hat{x}|[\hat{S}_z, \hat{S}_x]|+\hat{x}\rangle|$

$0 \geq 0$ works

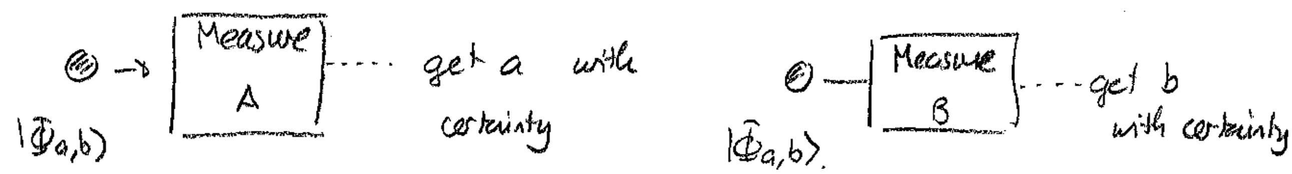
In this case the uncertainty principle does not help answer the question of compatibility since there is a special state $|\Psi\rangle$ such that all the uncertainty principle says is

$$\Delta S_z \Delta S_x \geq 0$$

But it does not say that both are zero. This can be answered definitely by considering the commutator. Again suppose that for two observables there is a state

$$|\Phi_{a,b}\rangle$$

such that



Then $|\Phi_{a,b}\rangle$ is an eigenstate of $\hat{A}$ and $\hat{B}$, So

$$\hat{A}|\Phi_{a,b}\rangle = a\,|\Phi_{a,b}\rangle$$

$$\hat{B}|\Phi_{a,b}\rangle = b\,|\Phi_{a,b}\rangle.$$

Thus

$$\begin{aligned}
[\hat{A},\hat{B}]\,|\Phi_{a,b}\rangle &= \hat{A}\hat{B}\,|\Phi_{a,b}\rangle - \hat{B}\hat{A}\,|\Phi_{a,b}\rangle \\
&= \hat{A}\,b\,|\Phi_{a,b}\rangle - \hat{B}\,a\,|\Phi_{a,b}\rangle \\
&= b\,\hat{A}\,|\Phi_{a,b}\rangle - a\,\hat{B}\,|\Phi_{a,b}\rangle \\
&= ba\,|\Phi_{a,b}\rangle - ab\,|\Phi_{a,b}\rangle = 0.
\end{aligned}$$

For the two measurements to be compatible, there must be such a state for each outcome of A and also each outcome of B. This means that there is a basis such that $[\hat{A},\hat{B}]=0$ for every basis element. Thus $[\hat{A},\hat{B}]=0$. A theorem of linear algebra guarantees that the converse is true. Thus

$$\boxed{\text{Measurements } A, B \text{ are compatible} \iff [\hat{A},\hat{B}]=0.}$$

For spin-1/2 particles, simple algebra gives:

$$[\hat{S}_x,\hat{S}_y] = i\hbar\,\hat{S}_z$$

$$[\hat{S}_y,\hat{S}_z] = i\hbar\,\hat{S}_x$$

$$[\hat{S}_z,\hat{S}_x] = i\hbar\,\hat{S}_y$$

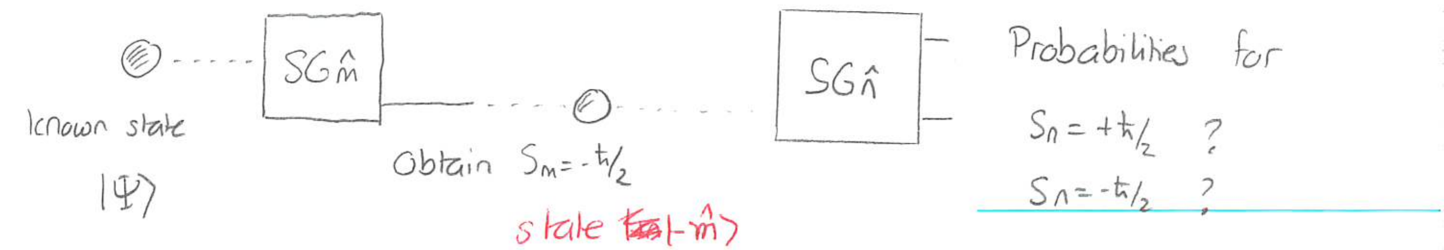
Thus none of the spin component measures S_x, S_y, S_z are compatible. On the other hand

$$[\hat{S}_x,\hat{S}_y^2] = 0$$

and thus S_x and S_y^2 are compatible.

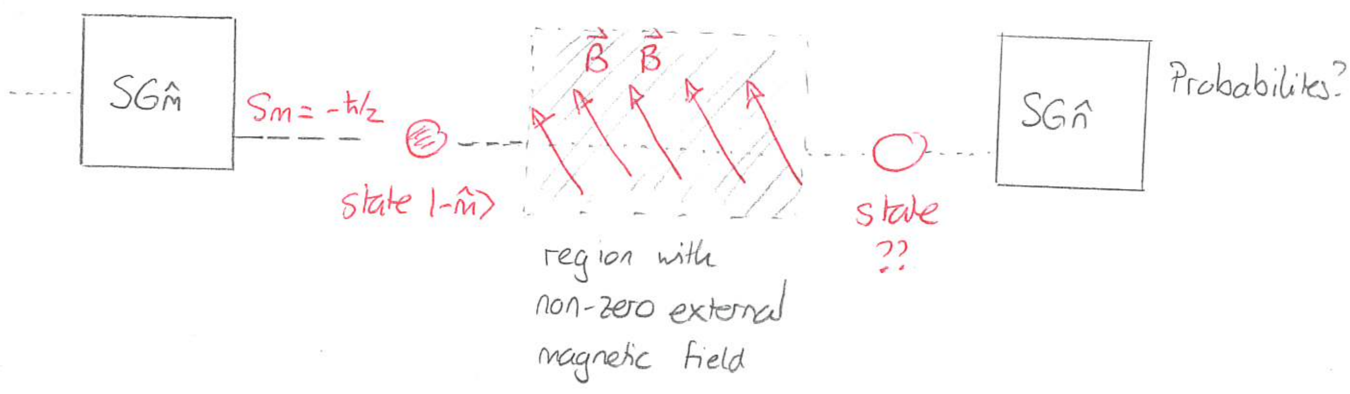
Evolution of quantum states

We have a framework for describing states and measurements and the relationship between these. So we can analyze situations such as



The analysis assumes that the state of the system between the two measurements stays fixed (although if $\hat{n} \neq \pm\hat{m}$ it will change after the second measurement). This will only be true if the particle does not interact with its surroundings during the period between the measurements.

However, it is possible that it could interact with its surroundings For example, there may be a region

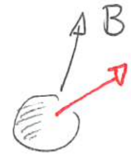


The presence of the external magnetic field will likely affect the probabilities of the subsequent measurement outcomes. Thus the presence of the field might change the state of the system:

Classical dipoles in a magnetic field

We can illustrate some of the basic notions via an analogous classical system: a magnetic dipole in an external magnetic field

In classical physics the rotational version of Newton's Second Law is:



$$\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

and for a spin angular momentum

$$\frac{d\vec{S}}{dt} = \vec{\tau}_{net}$$

Classical electromagnetism gives

Consider a magnetic dipole with dipole moment $\vec{\mu}$ in an external magnetic field $\vec{B}$. Then the torque on the dipole is

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Now $\vec{\mu} = \frac{gQ}{2M} \vec{S}$ and then

$$\frac{d\vec{\mu}}{dt} = \frac{gQ}{2M} \frac{d\vec{S}}{dt} = \frac{gQ}{2M} \vec{\tau}_{net} \Rightarrow \boxed{\frac{d\vec{\mu}}{dt} = \frac{gQ}{2M} \vec{\mu} \times \vec{B}}$$

Consider a special case of a constant magnetic field $\vec{B} = B_0 \hat{z}$

3 Classical dipole in a magnetic field

A magnetic dipole is placed in a constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$.

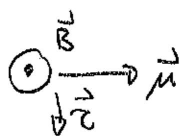
a) Suppose that $\boldsymbol{\mu}$ is initially perpendicular to $\mathbf{B}$. Describe qualitatively how $\boldsymbol{\mu}$ will evolve.

b) Determine a set of differential equations that the dipole moment components satisfy.

c) Show that the dipole moment components satisfy:

$$\mu_x(t) = \cos(\omega t)\mu_x(0) + \sin(\omega t)\mu_y(0)$$
$$\mu_y(t) = -\sin(\omega t)\mu_x(0) + \cos(\omega t)\mu_y(0)$$
$$\mu_z(t) = \mu_z(0)$$

Find an expression for ω in terms of q, Q, M and B_0.

Answer: a)  $\Rightarrow$ will rotate about $\vec{B}$.

b) $$\vec{\mu} = \mu_x\hat{x} + \mu_y\hat{y} + \mu_z\hat{z}$$

$$\Rightarrow \frac{d\vec{\mu}}{dt} = \frac{d\mu_x}{dt}\hat{x} + \frac{d\mu_y}{dt}\hat{y} + \frac{d\mu_z}{dt}\hat{z}$$

$$\vec{\mu}\times\vec{B} = \left[\mu_x\hat{x} + \mu_y\hat{y} + \mu_z\hat{z}\right]\times B_0\hat{z}$$

$$= \mu_x B_0 \underbrace{\hat{x}\times\hat{z}}_{-\hat{y}} + \mu_y B_0 \underbrace{\hat{y}\times\hat{z}}_{\hat{x}} + \cancel{\mu_z B_0 \hat{z}\times\hat{z}}$$

Thus

$$\frac{d\mu_x}{dt}\hat{x} + \frac{d\mu_y}{dt}\hat{y} + \frac{d\mu_z}{dt}\hat{z} = \left[\mu_y B_0\hat{x} - \mu_x B_0\hat{y}\right]\frac{gQ}{2M}.$$

$$\Rightarrow \frac{d\mu_x}{dt} = \frac{gQ}{2M}B_0\mu_y \qquad \frac{d\mu_z}{dt} = 0$$

$$\frac{d\mu_y}{dt} = -\frac{gQ}{2M}B_0\mu_x$$

c) With the given equations

$$\frac{d\mu_x}{dt} = -\omega \sin(\omega t)\, \mu_x(0) + \omega \cos(\omega t)\, \mu_y(0)$$

$$= \omega \mu_y$$

$$\frac{d\mu_y}{dt} = -\omega \cos(\omega t)\, \mu_x(0) - \omega \sin(\omega t)\, \mu_y(0)$$

$$= -\omega \mu_x$$

$$\frac{d\mu_z}{dt} = 0$$

These work provided

$$\omega = \frac{gQ}{2M} B_0$$

It follows that

$$\boxed{\text{In a magnetic field a dipole will precess (rotate) about the magnetic field}}$$

Note that the change in state can be expressed linearly

$$\underbrace{\begin{pmatrix} \mu_x(t) \\ \mu_y(t) \\ \mu_z(t) \end{pmatrix}}_{\text{state at later time}} = \underbrace{\begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{transformation operator}} \underbrace{\begin{pmatrix} \mu_x(0) \\ \mu_y(0) \\ \mu_z(0) \end{pmatrix}}_{\text{initial state } (t=0)}$$