Phys 321 2023

Lecture 8

Fri: HW 5pm

Tues: HW 5pm

Tues: Read. Text: 2.3, 2.4, 2.5

My notes: 3.4, 4.1

Observables

An observable is an operator that corresponds to a measurement. For a spin-1/2 particle these can be constructed from basic measurement ingredients via:

Physical measurement

Choose a direction $\hat{n}$ and measure component of spin along this direction

Mathematical description (basic ingredients)

Outcome	State	Measurement Operator
$S_n = +\hbar/2$	$\lvert +\hat{n}\rangle$	$\hat{P}_{+n} = \lvert +\hat{n}\rangle\langle +\hat{n}\rvert$
$S_n = -\hbar/2$	$\lvert -\hat{n}\rangle$	$\hat{P}_{-n} = \lvert -\hat{n}\rangle\langle -\hat{n}\rvert$

Describes probability of outcomes

$$\text{Prob}(S_n = +\hbar/2) = \langle\Psi|\hat{P}_{+n}|\Psi\rangle$$

$$\text{Prob}(S_n = -\hbar/2) = \langle\Psi|\hat{P}_{-n}|\Psi\rangle$$

Imagine ensemble of systems in same state subjected to same measurement. The outputs are aggregated into the sample average $\overline{S}_n$

Observable associated with S_n is

$$\hat{S}_n = +\frac{\hbar}{2}\hat{P}_{+n} - \frac{\hbar}{2}\hat{P}_{-n}$$

For particles in state $|\Psi\rangle$, mean, expectation value is

$$\langle S_n\rangle = \langle\Psi|\hat{S}_n|\Psi\rangle$$

These observables are Hermitian, i.e.

$$\boxed{\hat{S}_n^{\dagger} = \hat{S}_n}$$

Observables for spin-1/2 particles

For spin components, the observables have the form

$$\hat{S}_n = +\frac{\hbar}{2}\,|+\hat{n}\rangle\langle+\hat{n}| - \frac{\hbar}{2}\,|-\hat{n}\rangle\langle-\hat{n}|$$

$$= \frac{\hbar}{2}\left[\,|+\hat{n}\rangle\langle+\hat{n}| - |-\hat{n}\rangle\langle-\hat{n}|\,\right]$$

Along the three cardinal directions, direct calculation reveals that

$$\boxed{\hat{S}_x \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y \to \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z \to \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Then we can show that:

$\vec{n}$ For a given direction $\hat{n}$

In spherical co-ordinates:

$$\hat{n} = \cos\phi \sin\theta\, \hat{x} + \sin\phi \sin\theta\, \hat{y} + \cos\theta\, \hat{z}$$

Consider an ensemble of particles in state $|+\hat{n}\rangle$. Then for this ensemble

$$\langle S_x \rangle = \cos\phi \sin\theta \frac{\hbar}{2}$$

$$\langle S_y \rangle = \sin\phi \sin\theta \frac{\hbar}{2}$$

$$\langle S_z \rangle = \cos\theta \frac{\hbar}{2}$$

The expectation values are $\frac{\hbar}{2}$ times the components of $\hat{n}$

Similar algebra will give:

Given any unit vector

$$\hat{n} = \cos\phi \sin\theta \, \hat{x} + \sin\phi \sin\theta \, \hat{y} + \cos\theta \, \hat{z}$$

then the observable for S_n is

$$\hat{S}_n = \cos\phi \sin\theta \, \hat{\sigma}_x + \sin\phi \sin\theta \, \hat{\sigma}_y + \cos\theta \, \hat{\sigma}_z$$

Thus

Suppose

$$\hat{n} = n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$$

where

$$n_x^2 + n_y^2 + n_z^2 = 1$$

$\Rightarrow$

For any ensemble of particles in a given state

$$\langle S_n \rangle = n_x \langle S_x \rangle + n_y \langle S_y \rangle + n_z \langle S_z \rangle$$

Thus we only need to be able to determine $\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle$ in order to determine $\langle S_n \rangle$.

Eigenstates and eigenvalues of observables

We now consider the reverse process. Given a Hermitian operator, $\hat{A}$, can this be constructed as an observable? This means we need:

Outcomes	States	Measurement operators
a_1	$\|\phi_1\rangle$	$\hat{P}_1 = \|\phi_1\rangle\langle\phi_1\|$
a_2	$\|\phi_2\rangle$	$\hat{P}_2 = \|\phi_2\rangle\langle\phi_2\|$
a_3	$\|\phi_3\rangle$	$\hat{P}_3 = \|\phi_3\rangle\langle\phi_3\|$
$\vdots$	$\vdots$	$\vdots$

so that

$$\hat{A} = a_1\hat{P}_1 + a_2\hat{P}_2 + a_3\hat{P}_3 + \dots$$

In general this will be possible (although not necessarily unique). The process for finding the outcomes and states requires finding the eigenvalues and eigenstates of $\hat{A}$.

> Given an observable $\hat{A}$, an eigenstate $|\phi\rangle$ is a state that satisfies
>
> $$\hat{A}|\phi\rangle = \lambda|\phi\rangle$$
>
> where λ is a scalar.

1 Eigenstates of $\hat{S}_x$

Show that

$$\hat{S}_x \left|+\hat{x}\right\rangle = \lambda_1 \left|+\hat{x}\right\rangle$$

and

$$\hat{S}_x \left|-\hat{x}\right\rangle = \lambda_2 \left|-\hat{x}\right\rangle$$

and provide the values for λ_1 and λ_2.

Answer:

$$\hat{S}_x \sim \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|-\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{1}{\sqrt{2}}|-\hat{z}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus

$$\hat{S}_x|+\hat{x}\rangle \rightsquigarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\rightsquigarrow |+\hat{x}\rangle}$$

So

$$\hat{S}_x|+\hat{x}\rangle = \frac{\hbar}{2}|+\hat{x}\rangle \quad \Rightarrow \quad \lambda_1 = +h/2$$

Then

$$\hat{S}_x|-\hat{x}\rangle \rightsquigarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \hat{S}_x|-\hat{x}\rangle = -\frac{\hbar}{2}|-\hat{x}\rangle \quad \Rightarrow \quad \lambda_2 = -\hbar/2$$

This illustrates:

For the observable $\hat{S}_x$

eigenvalue	eigenstate
$+\hbar/2$	$\|+\hat{x}\rangle$
$-\hbar/2$	$\|-\hat{x}\rangle$

these match

Measurements of S_x

outcomes	associated state
$+\hbar/2$	$\|+\hat{x}\rangle$
$-\hbar/2$	$\|-\hat{x}\rangle$

This will generally be true for *any* observable.

Finding eigenstates and eigenvalues.

Consider an arbitrary operator $\hat{A}$. An eigenstate $|\phi\rangle$ satisfies

$$\hat{A}|\phi\rangle = \lambda|\phi\rangle = \lambda\hat{I}|\phi\rangle$$

$$\Rightarrow (\hat{A}-\lambda\hat{I})|\phi\rangle = 0$$

Solutions to this are either $|\phi\rangle = 0$ or $|\phi\rangle \neq 0$. In the latter case this requires that $\hat{A}-\lambda\hat{I}$ has no inverse. This is only possible if

$$\det(\hat{A}-\lambda\hat{I}) = 0$$

where the determinant of a 2×2 matrix is

$$\det\left[\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right] := b_{11}b_{22} - b_{12}b_{21}$$

Thus we have to:

Solve $\det(\hat{A}-\lambda\hat{I}) = 0$ for λ → Substitute λ into $\hat{A}|\phi\rangle = \lambda|\phi\rangle$ to determine $|\phi\rangle$ → Normalize $|\phi\rangle$

2 Eigenvalues of an observable

Consider

$$\hat{A} \leftrightarrow \frac{\hbar}{2}\frac{1}{5}\begin{pmatrix} 3 & -4i \\ 4i & -3 \end{pmatrix}.$$

a) Verify that $\hat{A}$ is Hermitian.

b) Find the eigenvalues and eigenstates of $\hat{A}$. Verify that each is an eigenstate using $\hat{A}\,|\phi\rangle = \lambda\,|\phi\rangle$.

c) Show that the two eigenstates are orthogonal.

Answer: a) $\hat{A}^{+} = \frac{\hbar}{2}\frac{1}{5}\begin{pmatrix} 3 & (4i)^* \\ (-4i)^* & -3 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{5}\begin{pmatrix} 3 & -4i \\ 4i & -3 \end{pmatrix} = \hat{A}$

b) Solve $\det(\hat{A}-\lambda\hat{I})=0$

$$\hat{A}-\lambda\hat{I} = \frac{\hbar}{2}\frac{1}{5}\begin{pmatrix} 3 & -4i \\ 4i & -3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\hbar}{2}\frac{3}{5}-\lambda & -\frac{\hbar}{2}\frac{4i}{5} \\ \frac{\hbar}{2}\frac{4i}{5} & -\frac{\hbar}{2}\frac{3}{5}-\lambda \end{pmatrix}$$

So

$$\det(\hat{A}-\lambda\hat{I}) = -\left(\frac{\hbar}{2}\frac{3}{5}-\lambda\right)\left(\frac{\hbar}{2}\frac{3}{5}+\lambda\right) - \frac{\hbar}{2}\frac{4i}{5}\left(\frac{\hbar}{2}\right)\left(-\frac{4i}{5}\right)$$

$$= -\left[\frac{\hbar^2}{4}\frac{9}{25}-\lambda^2\right] - \frac{\hbar^2}{4}\frac{16}{25}$$

$$= \lambda^2 - \frac{\hbar^2}{4}\frac{16+9}{25} = \lambda^2 - \frac{\hbar^2}{4}.$$

So $\det(\hat{A}-\lambda\hat{I})=0 \Leftrightarrow \lambda^2 = \frac{\hbar^2}{4} \Rightarrow \lambda = \pm\frac{\hbar}{2}$

Eigenvalue $+\hbar/2$ Let $|\phi\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$

$$\hat{A}|\phi\rangle = \frac{\hbar}{2}|\phi\rangle \Rightarrow \frac{\cancel{\hbar}}{\cancel{2}}\frac{1}{5}\begin{pmatrix} 3 & -4i \\ 4i & -3 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{\cancel{\hbar}}{\cancel{2}}\begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3c_+ - 4ic_- = 5c_+ \\ 4ic_+ - 3c_- = 5c_- \end{cases} \Rightarrow \begin{cases} -4ic_- = 2c_+ \\ 4ic_+ = 8c_- \end{cases}$$

$$\Rightarrow \quad c_- = \frac{i}{2}c_+$$

$$c_+ = -i2c_-$$

Thus

$$|\phi\rangle \sim \begin{pmatrix} c_+ \\ \frac{i}{2}c_+ \end{pmatrix} = c_+\begin{pmatrix} 1 \\ i/2 \end{pmatrix}$$

and

$$1 = \langle\phi|\phi\rangle = |c_+|^2(1 \ \ -i/2)\begin{pmatrix} 1 \\ i/2 \end{pmatrix} = |c_+|^2\left(\frac{5}{4}\right)$$

$$\Rightarrow \quad |c_+|^2 = \frac{4}{5} \Rightarrow |c_+| = \frac{2}{\sqrt{5}}$$

So a possibility is $c_+ = 2/\sqrt{5}$ and then

$$|\phi\rangle = \frac{2}{\sqrt{5}}\begin{pmatrix} 1 \\ i/2 \end{pmatrix} \Rightarrow |\phi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ i \end{pmatrix}$$

Eigenvalue $-\hbar/2$ Again let $|\phi\rangle \sim \begin{pmatrix} d_+ \\ d_- \end{pmatrix}$ so.

$$\frac{\cancel{\hbar}}{2}\frac{1}{5}\begin{pmatrix} 3 & -4i \\ 4i & -3 \end{pmatrix}\begin{pmatrix} d_+ \\ d_- \end{pmatrix} = -\frac{\cancel{\hbar}}{2}\begin{pmatrix} d_+ \\ d_- \end{pmatrix}$$

$$\Rightarrow \quad 3d_+ - 4id_- = -5d_+ \Rightarrow$$

$$\Rightarrow \quad 4id_+ - 3d_- = -5d_- \Rightarrow 2d_- = -4id_+ \Rightarrow \quad d_- = -2id_+$$

Thus $|\phi\rangle = d_+ \begin{pmatrix} 1 \\ -2i \end{pmatrix}$. Normalizing again gives: $d_+ = \frac{1}{\sqrt{5}}$.

Thus we have

eigenvalue	eigenstate
$+\hbar/2$	$\|\phi_1\rangle = \frac{1}{\sqrt{5}}\left[2\|+\hat{z}\rangle + i\|-\hat{z}\rangle\right]$
$-\hbar/2$	$\|\phi_2\rangle = \frac{1}{\sqrt{5}}\left[\|+\hat{z}\rangle - 2i\|-\hat{z}\rangle\right]$

Check.

$$\hat{A}|\phi_1\rangle \rightsquigarrow \frac{\hbar}{2}\frac{1}{5}\begin{pmatrix} 3 & -4i \\ 4i & -3 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ i \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{5}}\frac{1}{5}\begin{pmatrix} 10 \\ 5i \end{pmatrix} = \frac{\hbar}{2}\underbrace{\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ i \end{pmatrix}}_{|\phi_1\rangle}$$

$$\hat{A}|\phi_2\rangle \rightsquigarrow \frac{\hbar}{2}\frac{1}{5}\begin{pmatrix} 3 & -4i \\ 4i & -3 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2i \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{5}}\frac{1}{5}\begin{pmatrix} -5 \\ 10i \end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

c) $$\langle\phi_2|\phi_1\rangle = \frac{1}{\sqrt{5}}\left[\langle+\hat{z}| + 2i\langle-\hat{z}|\right]\frac{1}{\sqrt{5}}\left[2|+\hat{z}\rangle + i|-\hat{z}\rangle\right]$$

$$= \frac{1}{5}\left[2\underbrace{\langle+\hat{z}|+\hat{z}\rangle}_{1} + 4i\underbrace{\cancel{\langle-\hat{z}|+\hat{z}\rangle}}_{0} + i\underbrace{\cancel{\langle+\hat{z}|-\hat{z}\rangle}}_{0} - 2\underbrace{\langle-\hat{z}|-\hat{z}\rangle}_{1}\right] = 0.$$

Eigenstates and eigenvalues of observables.

Standard results from linear algebra give:

$$\boxed{\text{If } \hat{A} \text{ is Hermitian then any eigenvalue of } \hat{A} \text{ is real}}$$

> If $\hat{A}$ is Hermitian and λ_1, λ_2 are distinct eigenvalues with associated eigenstates $|\phi_1\rangle, |\phi_2\rangle$ then these are orthogonal.

In general

> If $\hat{A}$ is Hermitian then there exist real λ_i and a set of orthonormal states $|\phi_i\rangle$ such that
>
> $$\hat{A} = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$$
>
> Then $|\phi_i\rangle$ is an eigenstate of $\hat{A}$ with eigenvalue λ_i

It follows that $\hat{A}$ is constructed from measurement outcomes and states via:

$\hat{A}$ is Hermitian $\Rightarrow$

measurement outcome	associated state
λ_1	$\lvert\phi_1\rangle$
λ_2	$\lvert\phi_2\rangle$
$\vdots$	$\vdots$

This measurement results in the observable $\hat{A}$

Using the measurement operator construct

$$\hat{P}_i = |\phi_i\rangle\langle\phi_i|$$

gives

$$\hat{A} = \sum \lambda_i \hat{P}_i$$

outcome

associated measurement operator