Phys 321 2023

Lecture 7

Tues: HW 5pm

Thurs Read text 2.1 2.3-2.4

My notes 59-67

SPS noon WS 218

Fri: HW 5pm

## Measurement Operators

Recall that we can describe a measurement via outcomes and states. Thus for a $S_n$ measurement along direction $\hat{n}$:

| Outcome | Associated State | Measurement operator |
|---|---|---|
| $S_n = +\hbar/2$ | $\vert +\hat{n}\rangle$ | $\hat{P}_{+n} := \vert +\hat{n}\rangle\langle +\hat{n}\vert$ |
| $S_n = -\hbar/2$ | $\vert -\hat{n}\rangle$ | $\hat{P}_{-n} := \vert -\hat{n}\rangle\langle -\hat{n}\vert$ |

Then, for example if the state prior to measurement is $|\Psi\rangle$.

$$\text{Prob}(S_n = +\hbar/2) = |\langle +n|\Psi\rangle|^2$$

We can show that an alternative way to calculate this involves a measurement operator

$$\hat{P}_{+n} := |+\hat{n}\rangle\langle +\hat{n}|$$

and then

$$\text{Prob}(S_n = +\hbar/2) = \langle \Psi|\hat{P}_{+n}|\Psi\rangle$$

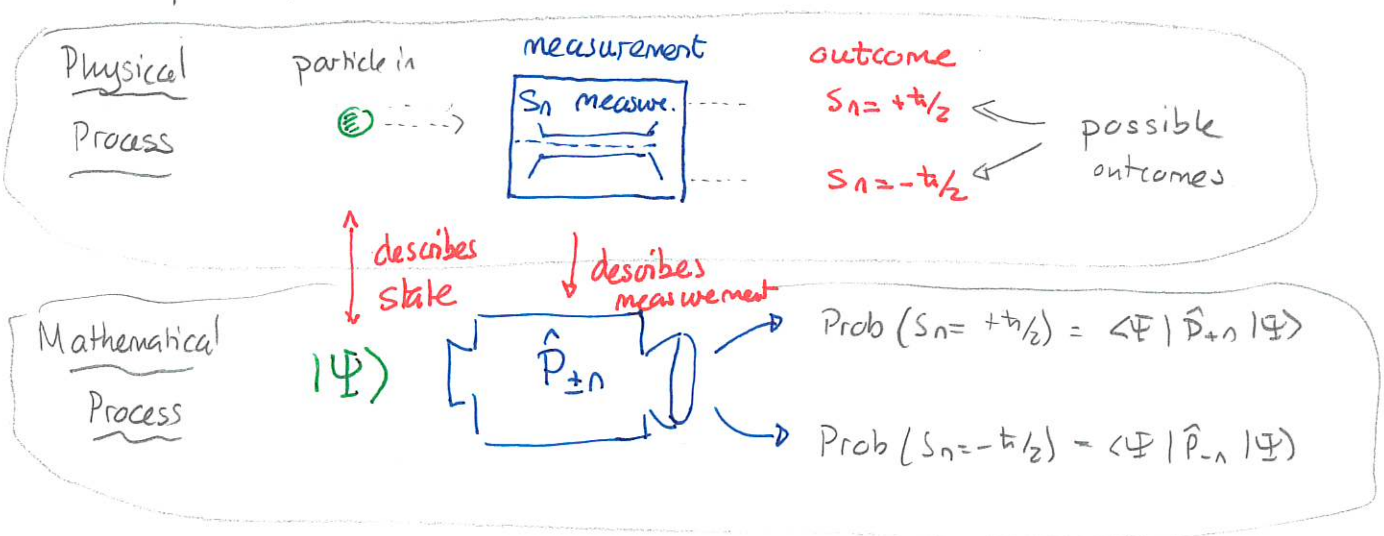
To demonstrate this,

$$\begin{aligned}
\text{Prob}(S_n = +\hbar/2) &= |\langle +\hat{n}|\Psi\rangle|^2 \\
&= \left(\langle +n|\Psi\rangle\right)^* \left(\langle +\hat{n}|\Psi\rangle\right) \\
&= \langle \Psi|+\hat{n}\rangle\langle +\hat{n}|\Psi\rangle \\
&= \langle \Psi|\left(|+\hat{n}\rangle\langle +\hat{n}|\right)|\Psi\rangle \qquad \blacksquare
\end{aligned}$$

Then in general

> Given a system in state $|\Psi\rangle$ prior to measurement then the probabilities with which the two outcomes occur are:
>
> $$\text{Prob}(S_n = +\hbar/2) = \langle \Psi|\hat{P}_{+n}|\Psi\rangle$$
>
> $$\text{Prob}(S_n = -\hbar/2) = \langle \Psi|\hat{P}_{-n}|\Psi\rangle$$

A conceptual picture is:



## 1 Measurement operators and probabilities

Consider a particle in the state

$$|+\hat{x}\rangle = \frac{1}{\sqrt{2}}\,|+\hat{z}\rangle + \frac{1}{\sqrt{2}}\,|-\hat{z}\rangle\,.$$

This is subjected to a measurement with the outcomes and associated states:

| | |
|---|---|
| Outcome: $+\hbar/2$ | State: $\lvert+\hat{n}\rangle = \frac{3}{5}\,\lvert+\hat{z}\rangle + \frac{4i}{5}\,\lvert-\hat{z}\rangle$ |
| Outcome: $-\hbar/2$ | State: $\lvert-\hat{n}\rangle = \frac{4}{5}\,\lvert+\hat{z}\rangle - \frac{3i}{5}\,\lvert-\hat{z}\rangle$ |

Construct the two measurement operators and use these to determine the probabilities of the two measurement outcomes for a particle in state $|+\hat{x}\rangle$.

Answer: $|+\hat{n}\rangle \leadsto \begin{pmatrix} 3/5 \\ \frac{4i}{5} \end{pmatrix} \Rightarrow \langle +\hat{n}| = \begin{pmatrix} \frac{3}{5} & -\frac{4i}{5} \end{pmatrix}$

$$\hat{P}_{+n} = \frac{1}{5}\begin{pmatrix} 3 \\ 4i \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 & -4i \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 9 & -12i \\ 12i & 16 \end{pmatrix} \Rightarrow \hat{P}_{+n} = \frac{1}{25}\begin{pmatrix} 9 & -12i \\ 12i & 16 \end{pmatrix}$$

$|-\hat{n}\rangle \leadsto \begin{pmatrix} 4/5 \\ -3i/5 \end{pmatrix} \Rightarrow \langle -\hat{n}| = \begin{pmatrix} \frac{4}{5} & \frac{3i}{5} \end{pmatrix}$

$$\hat{P}_{-n} = \frac{1}{5}\begin{pmatrix} 4 \\ -3i \end{pmatrix}\frac{1}{5}\begin{pmatrix} 4 & 3i \end{pmatrix} \Rightarrow \hat{P}_{-n} = \frac{1}{25}\begin{pmatrix} 16 & 12i \\ -12i & 9 \end{pmatrix} \Rightarrow \hat{P}_{-n} = \frac{1}{25}\begin{pmatrix} 16 & 12i \\ -12i & 9 \end{pmatrix}$$

$$\text{Prob}(S_n = +\hbar/2) = \langle +\hat{x}|\hat{P}_{+n}|+\hat{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\frac{1}{25}\begin{pmatrix} 9 & -12i \\ 12i & 16 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{50}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 9-12i \\ 12i+16 \end{pmatrix} = \frac{1}{50}(9-12i+12i+16) = \frac{1}{2}$$

$$\text{Prob}(S_n = -\hbar/2) = \langle +\hat{x}|\hat{P}_{-n}|+\hat{x}\rangle = \frac{1}{50}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 16 & 12i \\ -12i & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 16+12i \\ -12i+9 \end{pmatrix} = \frac{1}{2}$$

The general framework of quantum theory describes measurements in terms of operators like these. Then such operators have to satisfy certain general requirements:

1) the operators must return a positive outcome regardless of the state prior to measurement. Thus for any $|\Psi\rangle$

$$\langle \Psi | \hat{P}_{\pm n} | \Psi \rangle \geq 0$$

Any operator which satisfies this is called <u>positive</u>.

2) the probabilities must add to 1. This can be use to show that

$$\hat{P}_{+n} + \hat{P}_{-n} = \hat{I}$$

where $\hat{I}$ is the identity operator

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The fact that the operators are positive has an important and easily checked consequence described in terms of complex conjugation and transposition. To this end suppose that

$$\hat{A} \rightsquigarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

then the transpose of $\hat{A}$, denoted $\hat{A}^T$, is a matrix with an interchange of rows + colums. Thus

first row of $\hat{A}$ $\rightsquigarrow$ first column of $\hat{A}^T$

second " " $\hat{A}$ $\rightsquigarrow$ second " of $\hat{A}^T$

So

$$\hat{A}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

Properties of the transpose are:

1) $(\hat{A}+\hat{B})^T = \hat{A}^T + \hat{B}^T$
2) $(\lambda \hat{A})^T = \lambda \hat{A}^T$
3) $(\hat{A}\hat{B})^T = \hat{B}^T \hat{A}^T$ note order reversal
4) $(\hat{A}^T)^T = \hat{A}$

Then the adjoint or complex conjugate transpose is

$$\boxed{\hat{A}^\dagger := (\hat{A}^*)^T}$$

The properties of this are:

1) $(\hat{A}+\hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger$
2) $(\lambda\hat{A})^\dagger = \lambda^* \hat{A}^\dagger$
3) $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$ ← note order reversal
4) $(\hat{A}^\dagger)^\dagger = \hat{A}$

Then

$\boxed{\text{An operator, } \hat{A}, \text{ is Hermitian} \iff \hat{A}^\dagger = \hat{A}}$

Then a simple theorem states that any positive operator is Hermitian. Thus

$\boxed{\text{Any measurement operator is Hermitian.}}$

## 2 Measurement operators and probabilities

Consider a particle that is subjected to a measurement with the outcomes and associated states:

$$\text{Outcome: } +\hbar/2 \qquad \text{State: } |+\hat{\boldsymbol{n}}\rangle = \frac{3}{5}\,|+\hat{\boldsymbol{z}}\rangle + \frac{4i}{5}\,|-\hat{\boldsymbol{z}}\rangle$$

$$\text{Outcome: } -\hbar/2 \qquad \text{State: } |-\hat{\boldsymbol{n}}\rangle = \frac{4}{5}\,|+\hat{\boldsymbol{z}}\rangle - \frac{3i}{5}\,|-\hat{\boldsymbol{z}}\rangle$$

Construct the two measurement operators and verify that they are Hermitian and add to the identity.

$$\hat{P}_{+n} = |+\hat{n}\rangle\langle+\hat{n}| = \frac{1}{25}\begin{pmatrix}3\\4i\end{pmatrix}\begin{pmatrix}3 & -4i\end{pmatrix} = \frac{1}{25}\begin{pmatrix}9 & -12i\\ 12i & 16\end{pmatrix}$$

$$\hat{P}_{-n} = |-\hat{n}\rangle\langle-\hat{n}| = \frac{1}{25}\begin{pmatrix}4\\-3i\end{pmatrix}\begin{pmatrix}4 & 3i\end{pmatrix} = \frac{1}{25}\begin{pmatrix}16 & 12i\\ -12i & 9\end{pmatrix}$$

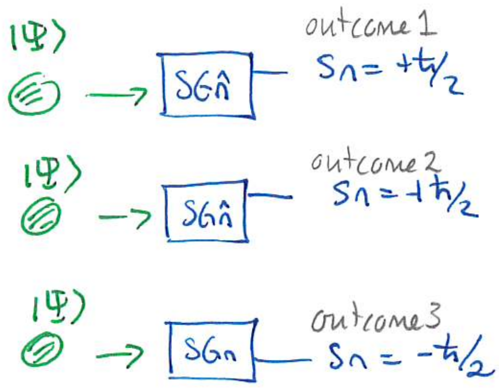
So $\hat{P}_{+n}^{\dagger} = \frac{1}{25}\begin{pmatrix}9 & (12i)^*\\ (-12i)^* & 16\end{pmatrix} = \frac{1}{25}\begin{pmatrix}9 & -12i\\ 12i & 16\end{pmatrix} \Rightarrow \hat{P}_{+n}^{\dagger} = P_{+n}$

Then $\hat{P}_{-n}^{\dagger} = \frac{1}{25}\begin{pmatrix}16 & (-12i)^*\\ (+12i)^* & 9\end{pmatrix} = \frac{1}{25}\begin{pmatrix}16 & +12i\\ -12i & 9\end{pmatrix} \Rightarrow \hat{P}_{-n}^{\dagger} = \hat{P}_{-n}$

$$\hat{P}_{+n} + \hat{P}_{-n} = \frac{1}{25}\begin{pmatrix}9+16 & -12i+12i\\ 12i-12i & 16+9\end{pmatrix} = \frac{1}{25}\begin{pmatrix}25 & 0\\ 0 & 25\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \hat{I}$$

# Expectation values and observables

In many situations multiple independent quantum systems are subjected to the same measurements. Even if the systems were all in the same state, one could get different outcomes. Suppose that the



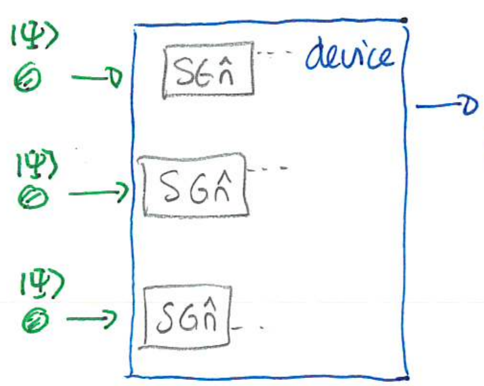

measurement outcomes are collected by the devices and aggregated into a single sample average

$$\frac{\text{outcome 1} + \text{outcome 2} + \text{outcome 3} + \dots}{\text{number of particles}}$$

Can we predict the sample average? If the sample is large enough then the sample average will be very close to the expectation value (with high probability). The expectation value can be used to predict the aggregated outcomes of such experiments. It is

$$\boxed{\langle S_n \rangle := \frac{+\hbar}{2}\,\mathrm{Prob}(S_n = +\hbar/2) + \left(-\frac{\hbar}{2}\right)\mathrm{Prob}(S_n = -\hbar/2)}$$

For example




outputs one number consisting of sum of all individual outputs $\approx$ number of systems $\times \langle S_n \rangle$

Then

$$\langle S_n \rangle = \frac{+\hbar}{2} \langle \Psi | \hat{P}_{+n} | \Psi \rangle - \frac{\hbar}{2} \langle \Psi | \hat{P}_{n} | \Psi \rangle$$

$$= \frac{\hbar}{2} \langle \Psi | \left[ \hat{P}_{+n} - \hat{P}_{-n} \right] | \Psi \rangle$$

$$= \langle \Psi | \frac{\hbar}{2} \left[ \hat{P}_{+n} - \hat{P}_{-n} \right] | \Psi \rangle.$$

Thus we define a single operator associated with the $S_n$ measurement, called the $\hat{S}_n$ observable:

$$\boxed{\hat{S}_n := \frac{\hbar}{2} \left( \hat{P}_{+n} - \hat{P}_{-n} \right)}$$

So

For a system in state $|\Psi\rangle$,

$$\langle S_n \rangle = \langle \Psi | \hat{S}_n | \Psi \rangle$$

It is straightforward to show that

$$\boxed{\hat{S}_n^{+} = \hat{S}_n}$$

and so $\hat{S}_n$ is Hermition

## 3 $S_z$ observable

a) Determine the matrix representation for $\hat{S}_z$.

b) Verify that this is Hermitian.

c) Suppose that a collection of particles are each in the state

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|+\hat{z}\rangle + \sin\left(\frac{\theta}{2}\right)|-\hat{z}\rangle .$$

Determine the expectation value for this ensemble.

Ans: a) $\hat{S}_z = +\frac{\hbar}{2}|+\hat{z}\rangle\langle+\hat{z}| - \frac{\hbar}{2}|-\hat{z}\rangle\langle-\hat{z}|$

$$= +\frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

b) $\hat{S}_z^{+} = \left(\frac{\hbar}{2}\right)^{*}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{+} = \left(\frac{\hbar}{2}\right)\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

c) $\langle S_z\rangle = \langle\Psi|\hat{S}_z|\Psi\rangle$

$$= \begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\cos\theta/2\\ \sin\theta/2\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta/2 & \sin\theta/2\end{pmatrix}\begin{pmatrix}\cos\theta/2\\ -\sin\theta/2\end{pmatrix} = \frac{\hbar}{2}\underbrace{\left(\cos^2\theta/2 - \sin^2\theta/2\right)}_{\cos\theta}$$

$$\Rightarrow \langle S_z\rangle = \frac{\hbar}{2}\cos\theta$$