Phys 321 2023

Lecture 6

Fri: HW 5pm

Tues: Read 2.2, 2.3

My notes: Section 3.3

HW

Bra vectors

Recall that a ket is represented by a column vector. Thus

$$|\Psi\rangle = b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle \leadsto \begin{pmatrix} b_+ \\ b_- \end{pmatrix}$$

A bra vector is a row vector and can be constructed from a ket via

$$\text{ket} \quad |\Phi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle \leadsto \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \longrightarrow \text{bra} \quad \langle\Phi| \leadsto \begin{pmatrix} a_+^* & a_-^* \end{pmatrix}$$

Then a bra acts on a ket via matrix multiplication. This has the effect.

Given kets $|\Psi\rangle, |\Phi\rangle$ $\longrightarrow$ Construct bra $\langle\Phi|$

Bra acting on ket gives inner product of original kets $\quad \langle\Phi||\Psi\rangle = \langle\Phi|\Psi\rangle$

Then using the special bra vectors $\langle +\hat{z}| \leadsto (1 \; 0)$, $\langle -\hat{z}| \leadsto (0 \; 1)$ gives

$$\boxed{\text{If } |\Phi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle \text{ then } \langle\Phi| = a_+^* \langle +\hat{z}| + a_-^* \langle -\hat{z}|}$$

Fundamental rules are

$$\langle +\hat{z} | +\hat{z} \rangle = 1$$

$$\langle +\hat{z} | -\hat{z} \rangle = 0$$

$$\langle -\hat{z} | +\hat{z} \rangle = 0$$

$$\langle -\hat{z} | -\hat{z} \rangle = 1$$

The algebra of the bra on ket operations can be simplified by their linearity.

$$\boxed{\begin{array}{l} \text{If } \quad |\Psi\rangle = \alpha_1 |\Psi_1\rangle + \alpha_2 |\Psi_2\rangle \quad \text{then} \\[1em] \quad \langle \Phi | \Psi \rangle = \alpha_1 \langle \Phi | \Psi_1 \rangle + \alpha_2 \langle \Phi | \Psi_2 \rangle \\[2em] \text{If } \quad \langle \Phi | = \beta_1 \langle \Phi_1 | + \beta_2 \langle \Phi_2 | \quad \text{then} \\[1em] \quad \langle \Phi | \Psi \rangle = \beta_1 \langle \Phi_1 | \Psi \rangle + \beta_2 \langle \Phi_2 | \Psi \rangle \end{array}}$$

We can evaluate the action of a bra on a ket using these linearity rules (that appear like ordinary algebra rules) and the basic bra/ket operations above.

1 Bra vectors

Let

$$|\Phi\rangle = \frac{1}{\sqrt{5}}\,|+\hat{z}\rangle + \frac{2-2i}{\sqrt{10}}\,|-\hat{z}\rangle .$$

a) Express $\langle\Phi|$ as a row vector and use this to express $\langle\Phi|$ as a linear combination of $\langle+\hat{z}|$ and $\langle-\hat{z}|$.

b) Let

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,|+\hat{z}\rangle - \frac{i}{\sqrt{2}}\,|-\hat{z}\rangle .$$

Use the bra/ket notation to determine $\langle\Phi|\Psi\rangle$.

Answer: a) $|\Phi\rangle \rightsquigarrow \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2-2i}{\sqrt{10}} \end{pmatrix} \Rightarrow \langle\Phi| \rightsquigarrow \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2+2i}{\sqrt{10}} \end{pmatrix}$

Now $\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2+2i}{\sqrt{10}} \end{pmatrix} = \frac{1}{\sqrt{5}}(1\ 0) + \frac{2+2i}{\sqrt{10}}(0\ 1)$

$\rightsquigarrow \frac{1}{\sqrt{5}}\langle+\hat{z}| + \frac{2+2i}{\sqrt{10}}\langle-\hat{z}|$

$\Rightarrow \langle\Phi| = \frac{1}{\sqrt{5}}\langle+\hat{z}| + \frac{2+2i}{\sqrt{10}}\langle-\hat{z}|$

b) $\langle\Phi|\Psi\rangle = \left[\frac{1}{\sqrt{5}}\langle+\hat{z}| + \frac{2+2i}{\sqrt{10}}\langle-\hat{z}|\right]\left[\frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle\right]$

$= \frac{1}{\sqrt{10}}\underbrace{\cancel{\langle+\hat{z}|+\hat{z}\rangle}}_{1} + \frac{2+2i}{\sqrt{20}}\underbrace{\cancel{\langle-\hat{z}|+\hat{z}\rangle}}_{0} - \frac{i}{\sqrt{10}}\underbrace{\cancel{\langle+\hat{z}|-\hat{z}\rangle}}_{0} - \frac{i(2+2i)}{\sqrt{20}}\underbrace{\cancel{\langle-\hat{z}|-\hat{z}\rangle}}_{1}$

$= \frac{1}{\sqrt{10}} - \frac{i(1+i)}{\sqrt{5}} = \frac{1}{\sqrt{10}} + \frac{1-i}{\sqrt{5}}$

$= \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{5}} - \frac{i}{\sqrt{5}}$

We can further automate this process via an algebraic operation that transforms between row and column vectors. The transpose operation is:

column $\to$ row

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}^T \to \begin{pmatrix} a_1 & a_2 & a_3 & \cdots \end{pmatrix}$$

transpose

row $\to$ column

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots \end{pmatrix}^T \to \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}$$

The adjoint operation or the complex conjugate transpose is the transpose together with complex conjugation, denoted with a dagger $\dagger$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}^\dagger = \begin{pmatrix} a_1^* & a_2^* & a_3^* & \cdots \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 & \cdots \end{pmatrix}^\dagger = \begin{pmatrix} a_1^* \\ a_2^* \\ a_3^* \\ \vdots \end{pmatrix}$$

It follows that

$$|+\hat{z}\rangle^\dagger = \langle +\hat{z}|$$
$$|-\hat{z}\rangle^\dagger = \langle -\hat{z}|$$
$$\langle +\hat{z}|^\dagger = |+\hat{z}\rangle$$
$$\langle -\hat{z}|^\dagger = |-\hat{z}\rangle$$

In general one can show

$$\boxed{\begin{aligned} \left[\alpha_1|\Psi_1\rangle + \alpha_2|\Psi_2\rangle\right]^\dagger &= \alpha_1^*\,|\Psi_1\rangle^\dagger + \alpha_2^*\,|\Psi_2\rangle^\dagger = \alpha_1^*\langle\Psi_1| + \alpha_2^*\langle\Psi_2| \\ \left[\beta_1\langle\Phi_1| + \beta_2\langle\Phi_2|\right]^\dagger &= \beta_1^*\,\langle\Phi_1|^\dagger + \beta_2^*\,\langle\Phi_2|^\dagger = \beta_1^*\,|\Phi_1\rangle + \beta_2^*\,|\Phi_2\rangle \end{aligned}}$$

and

$$\boxed{\begin{aligned} \left[|\Psi\rangle^\dagger\right]^\dagger &= |\Psi\rangle \\ \left[\langle\Phi|^\dagger\right]^\dagger &= \langle\Phi| \end{aligned}}$$

2 Measurements and bra/ket vectors

A spin-1/2 particle is in the state $|+\hat{x}\rangle$ and is subjected to an SG $\hat{n}$ measurement along the direction $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}\hat{\mathbf{x}} + \frac{1}{\sqrt{2}}\hat{\mathbf{y}}$. Use the bra/ket formalism to determine the probabilities of the measurement outcomes.

Answer: $\text{Prob}(S_n = +\hbar/2) = |\langle +\hat{n}|+\hat{x}\rangle|^2$

$\text{Prob}(S_n = -\hbar/2) = |\langle -\hat{n}|+\hat{x}\rangle|^2$

Now we need to

1) construct $|+\hat{x}\rangle$
2) construct $|\pm\hat{n}\rangle \rightsquigarrow$ use these to construct $\langle \pm\hat{n}|$
3) act with $\langle \pm\hat{n}|$ on $|+\hat{x}\rangle$

For $|+\hat{x}\rangle$ $\quad \theta = \pi/2 \quad \phi = 0 \quad \Rightarrow |+x\rangle = \cos\frac{\pi}{4}|+\hat{z}\rangle + e^{i0}\sin\frac{\pi}{4}|-\hat{z}\rangle$

$$\Rightarrow |+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$$

For $|\pm\hat{n}\rangle$ $\quad \theta = \pi/2 \quad \phi = \pi/4 \quad \Rightarrow |+\hat{n}\rangle = \cos\frac{\pi}{4}|+\hat{z}\rangle + e^{i\pi/4}\sin\frac{\pi}{4}|-\hat{z}\rangle$

$|-\hat{n}\rangle = \sin\frac{\pi}{4}|+\hat{z}\rangle - e^{i\pi/4}\cos\frac{\pi}{4}|-\hat{z}\rangle$

$$\Rightarrow \begin{aligned} |+\hat{n}\rangle &= \frac{1}{\sqrt{2}}|+\hat{z}\rangle + e^{i\pi/4}\frac{1}{\sqrt{2}}|-\hat{z}\rangle \\ |-\hat{n}\rangle &= \frac{1}{\sqrt{2}}|+\hat{z}\rangle - e^{i\pi/4}\frac{1}{\sqrt{2}}|-\hat{z}\rangle \end{aligned}$$

Now $\langle +\hat{n}| = |+\hat{n}\rangle^\dagger = \left[\frac{1}{\sqrt{2}}|+\hat{z}\rangle + e^{i\pi/4}\frac{1}{\sqrt{2}}|-\hat{z}\rangle\right]^\dagger = \left(\frac{1}{\sqrt{2}}\right)^* |+\hat{z}\rangle^\dagger + \left(e^{i\pi/4}\frac{1}{\sqrt{2}}\right)^* |-\hat{z}\rangle^\dagger$

$$\Rightarrow \langle +\hat{n}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| + e^{-i\pi/4}\frac{1}{\sqrt{2}}\langle -\hat{z}|.$$

Similarly

$$\langle -\hat{n}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| - e^{-i\pi/4}\frac{1}{\sqrt{2}}\langle -\hat{z}|$$

So $$\langle +\hat{n}|+\hat{x}\rangle = \left[\frac{1}{\sqrt{2}}\langle +\hat{z}| + e^{-i\pi/4}\frac{1}{\sqrt{2}}\langle -\hat{z}|\right]\left[\frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle\right]$$

$$= \frac{1}{2}\left[\langle +\hat{z}| + e^{-i\pi/4}\langle -\hat{z}|\right]\left[|+\hat{z}\rangle + |-\hat{z}\rangle\right]$$

$$= \frac{1}{2}\left[1 + e^{-i\pi/4}\right]$$

similarly $\langle -\hat{n}|+\hat{x}\rangle = \frac{1}{2}\left[1 - e^{-i\pi/4}\right]$.

So $$|\langle +\hat{n}|+\hat{x}\rangle|^2 = \left[\frac{1}{2}(1+e^{-i\pi/4})\right]\left[\frac{1}{2}(1+e^{-i\pi/4})\right]^*$$

$$= \frac{1}{4}\left(1+1+e^{-i\pi/4}+e^{i\pi/4}\right)$$

$$= \frac{1}{4}\left(2+2\cos\pi/4\right) = \frac{1}{2}\left(1+\cos\pi/4\right) = \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)$$

Similarly

$$|\langle -\hat{n}|+\hat{x}\rangle|^2 = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$$

So

$$\text{Prob}\left(S_n = +\hbar/2\right) = \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)$$

$$\text{Prob}\left(S_n = -\hbar/2\right) = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$$

Operators

So far the mathematics for quantum systems is

States of the system represented by kets

$$|\Psi\rangle \rightsquigarrow \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

column vectors

There is an inner product between any two kets. This produces a complex number

Inner products are used to determine the probabilities of measurement outcomes

Bra vectors are row vectors constructed from ket vectors

$$\langle \Psi | = |\Psi\rangle^\dagger$$

$$\rightsquigarrow (a_+^* \; a_-^*)$$

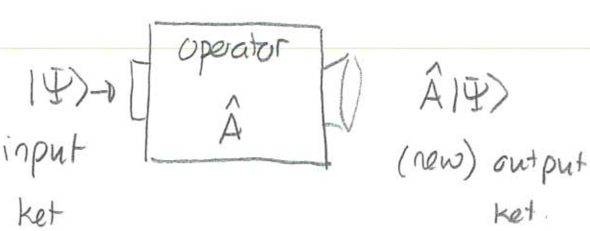
Bra vectors act on ket vectors and produce the inner product result:

$$\langle \Phi | \Psi \rangle$$

We now introduce a new mathematical ingredient - operators (matrices). These typically act on kets to produce new kets. In quantum theory these appear in:

1) descriptions of evolution processes - how a system evolves with time.
2) " " measurements ~ the most general description of measurements involves particular operators
3) " " states - the most general states are described via quantum operators.

In general an operator, denoted $\hat{A}$, maps kets to kets:



For kets corresponding to spin-½ particles operators can be represented via matrices.

$$\hat{A} \rightsquigarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then for a ket

$$|\Psi\rangle = b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle \rightsquigarrow \begin{pmatrix} b_+ \\ b_- \end{pmatrix}$$

$$\hat{A}|\Psi\rangle \rightsquigarrow \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\substack{2\times 2 \\ \text{matrix}}} \underbrace{\begin{pmatrix} b_+ \\ b_- \end{pmatrix}}_{\substack{\text{Column} \\ \text{vector}}} = \underbrace{\begin{pmatrix} a_{11} b_+ + a_{12} b_- \\ a_{21} b_+ + a_{22} b_- \end{pmatrix}}_{\text{column vector}}$$

$$\hookrightarrow (a_{11} b_+ + a_{12} b_-) |+\hat{z}\rangle + (a_{21} b_+ + a_{22} b_-) |-\hat{z}\rangle$$

A key aspect of such operators is that they are linear. So

$$\boxed{\hat{A}\left[\alpha_1 |\Psi_1\rangle + \alpha_2 |\Psi_2\rangle\right] = \alpha_1 \hat{A}|\Psi_1\rangle + \alpha_2 \hat{A}|\Psi_2\rangle}$$

3 Operator action on kets

Suppose that (in the $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ basis),

$$\hat{A} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4i & 3i \end{pmatrix}.$$

a) Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{1}{\sqrt{2}}|-\hat{z}\rangle$$

Determine an expression for $\hat{A}|\psi\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

b) Determine all four possibilities for

$$\langle \pm\hat{\mathbf{z}}| \left(\hat{A}|\pm\hat{\mathbf{z}}\rangle\right)$$

How could these be used to reconstruct the operator?

Answer: a) $|\Psi\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\hat{A}|\Psi\rangle = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4i & 3i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{5\sqrt{2}}\begin{pmatrix} 3+4 \\ 4i-3i \end{pmatrix} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 \\ i \end{pmatrix}$$

$$= \frac{7}{5\sqrt{2}}|+\hat{z}\rangle + \frac{i}{5\sqrt{2}}|-\hat{z}\rangle$$

b) $\hat{A}|+\hat{z}\rangle \rightsquigarrow \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4i & 3i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 4i \end{pmatrix} \Rightarrow \langle +\hat{z}|\hat{A}|+\hat{z}\rangle = (1\ 0)\frac{1}{5}\begin{pmatrix} 3 \\ 4i \end{pmatrix} = \frac{3}{5}$

$\Rightarrow \langle -\hat{z}|\hat{A}|+\hat{z}\rangle = (0\ 1)\frac{1}{5}\begin{pmatrix} 3 \\ 4i \end{pmatrix} = \frac{4i}{5}$

$\hat{A}|-\hat{z}\rangle \rightsquigarrow \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4i & 3i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 \\ 3i \end{pmatrix} \Rightarrow \langle +\hat{z}|\hat{A}|-\hat{z}\rangle = (1\ 0)\frac{1}{5}\begin{pmatrix} -4 \\ 3i \end{pmatrix} = -\frac{4}{5}$

$\Rightarrow \langle -\hat{z}|\hat{A}|-\hat{z}\rangle = (0\ 1)\frac{1}{5}\begin{pmatrix} -4 \\ 3i \end{pmatrix} = \frac{3i}{5}$

so $$\hat{A} = \begin{pmatrix} \langle +\hat{z}|\hat{A}|+\hat{z}\rangle & \langle -\hat{z}|\hat{A}|+\hat{z}\rangle \\ \langle +\hat{z}|\hat{A}|-\hat{z}\rangle & \langle -\hat{z}|\hat{A}|-\hat{z}\rangle \end{pmatrix}$$

Bra / ket operator construction

It is possible to construct an operator via a ket / bra product. Let

$$|\Psi\rangle = a_1 |+\hat{z}\rangle + a_2 |-\hat{z}\rangle \leftrightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\langle\Phi| = b_1 \langle+\hat{z}| + b_2 \langle-\hat{z}| \leftrightarrow \begin{pmatrix} b_1 & b_2 \end{pmatrix}$$

Then consider an "outer" product of the form:

$$|\Psi\rangle\langle\Phi|.$$

This acts as an operator as follows

$$\underbrace{|\chi\rangle}_{\text{ket}} \quad \boxed{\begin{matrix}\text{operator} \\ |\Psi\rangle\langle\Phi|\end{matrix}} \quad \underbrace{\underbrace{|\Psi\rangle}_{\text{ket}}\underbrace{(\langle\Phi|\chi\rangle)}_{\text{complex number}}}_{\text{ket}}$$

In terms of matrices:

$$|\Psi\rangle\langle\Phi| \leftrightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

4 Ket/bra operator construction

a) Construct the matrix for $|+\hat{z}\rangle \langle +\hat{z}|$.

b) Construct the matrix for $|+\hat{z}\rangle \langle -\hat{z}|$.

c) Construct the matrix for $|-\hat{z}\rangle \langle +\hat{z}|$.

d) Construct the matrix for $|-\hat{z}\rangle \langle -\hat{z}|$.

Answer: a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$