Phys 321 2023

Lecture 5

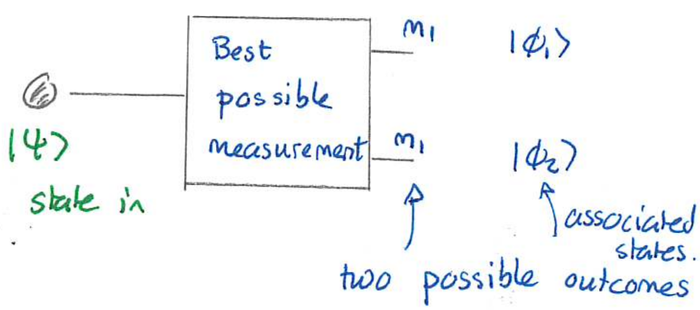
Tues: HW by 5pm

Thurs: Read 2.1, 2.2
My notes pg 40 – 43, Section 3.3

Fri: HW by 5pm

Two state systems

The most detailed measurements for a spin-$\frac{1}{2}$ system are the spin component measurements. These always yield one of two possible outcomes. A system for which this is true is called a two-state system (in quantum information, a qubit). The two states associated with the outcomes of a single measurement form an orthonormal basis. Thus



$$|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle \sim \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

complex

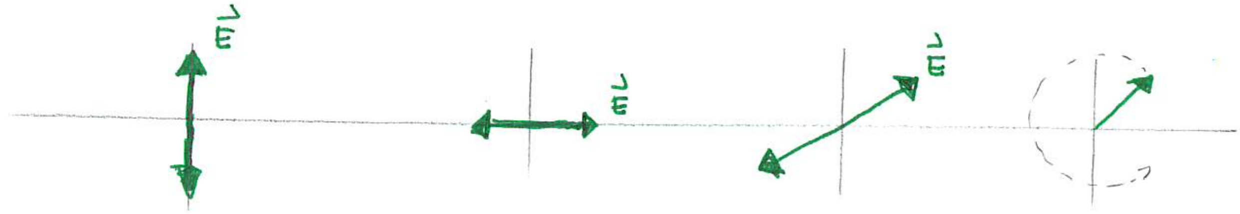
Examples of such systems include:

1) photon polarization
2) photon which way in two-arm interferometers
3) ammonia molecule — Feynman Vol III
4) neutrino oscillations. — Sassaroli AJP 67, 869 (1999)
   Wolfenstein, Rev Mod Phys, 71, S140 (1999)

The mathematics of all such systems is identical to that of spin-$\frac{1}{2}$ particles. It's just the nature of the measurements, interactions and evolutions that differ.

Photon polarization

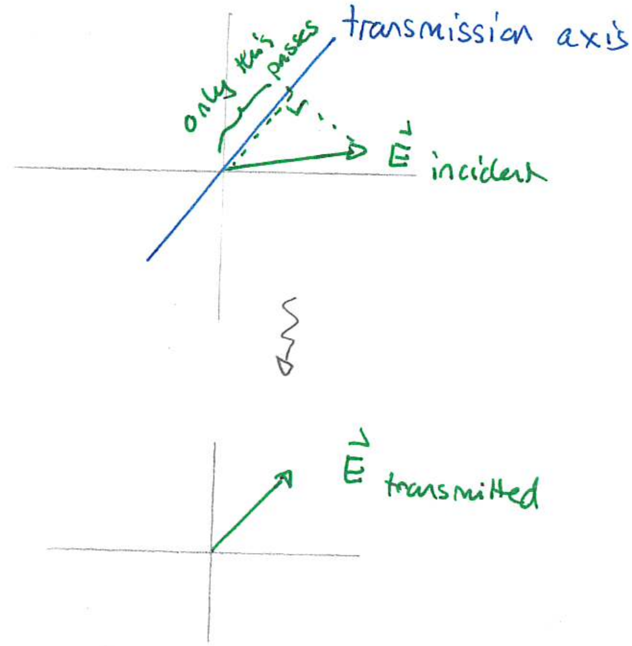
An example of a two-state system is the polarization of a single photon. We first provide a classical description of polarization, which considers the direction of the electric field vector as a wave propagates. Consider the wave passing a single location



always vertical	always horizontal	always one line	rotates
vertically polarized	horizontally polarized	linearly pol.	circularly pol.

These can be created and analyzed using a polarization filter. This allows the electric field component along one special axis – the polarizer transmission axis – to pass.

Typically this will reduce the size of the transmitted electric field vector and therefore the intensity of the transmitted light.



Note that a general rule for the intensity of (sinusoidal wave) light is

$$I = \frac{1}{2} c \epsilon_0 |\vec{E}|^2$$

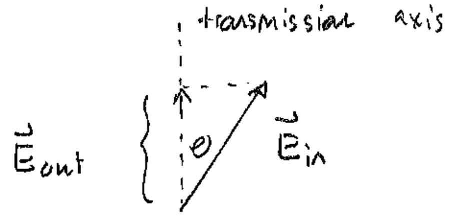
where c = speed of light

ϵ_0 = permittivity of free space

1 Malus' Law

Classical linearly polarized light is incident on a linear polarization filter whose. The angle between the line of polarization of the incident electric field and the transmission axis of the filter is θ. Let I_{in} be the intensity of the incident light and I_{out} the intensity of the transmitted light. Determine a relationship between I_{out} and I_{in} in terms of θ.

Answer:



$$E_{out} = \cos\theta \, E_{in}$$

$$I_{out} = \frac{1}{2} c\epsilon_0 E_{out}^2$$

$$= \frac{1}{2} c\epsilon_0 \cos^2\theta \, E_{in}^2$$

$$= \cos^2\theta \underbrace{\frac{1}{2} c\epsilon_0 E_{in}^2}_{I_{in}} \Rightarrow I_{out} = \cos^2\theta \, I_{in}$$

This is Malus' Law. Thus in special cases:

1) transmission axis parallel to incident direction of polarization

$\Rightarrow I_{out} = I_{in}$

2) transmission axis perpendicular to incident direction of polarization

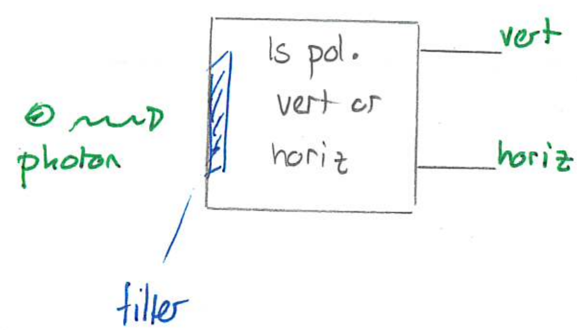
$\Rightarrow I_{out} = 0$

3) transmission axis at 45° $\Rightarrow I_{out} = \frac{1}{2} I_{in}$

Single photons of light can also have polarization even though they are not described using classical waves. For such photons

1) the intensity of any detected light is proportional to the number of photons detected per second.

2) a polarizing filter can measure the polarization state of a photon. For example

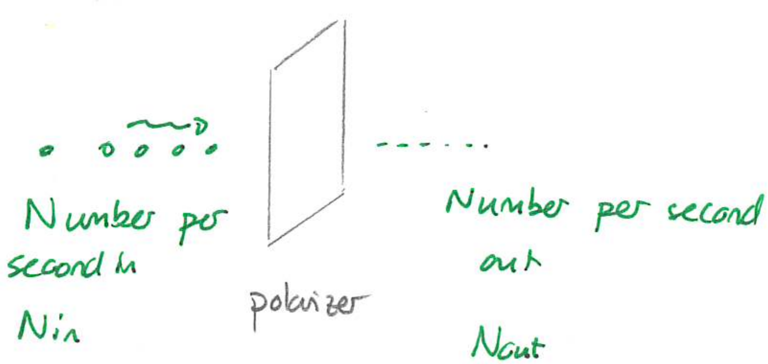


if transmitted state is vert

if blocked state is horiz

It does so by checking whether a photon is transmitted or not.

3) Malus' Law applies with intensities replaced by photon numbers



If linearly polarized photons are incident and the angle between the transmission axis and polarization direction is θ then

$$N_{out} = \cos^2\theta \; N_{in}$$

2 Polarizing filter as a measuring device

Consider a linear polarizing filter with vertical transmission axis as a measuring device when single photons are incident.

a) Suppose that a photon is transmitted. If the measurement were repeated what would the result be? How could one describe the state of this photon.

b) Consider a photon whose polarization was perpendicular to the transmission axis. Would the filter measurement produce one outcome with certainty? How could one describe the state of the photon prior to the measurement?

It should be clear that there are two states: vertically polarized, $|\updownarrow\rangle$, and horizontally polarized $|\leftrightarrow\rangle$. The state $|\updownarrow\rangle$ means that if a photon is in that state and is subjected to a measurement of its vertical versus horizontal polarization, the outcome will definitely be vertical. A similar statement applies to the state $|\leftrightarrow\rangle$.

c) What special mathematical property do the states $\{|\updownarrow\rangle, |\leftrightarrow\rangle\}$ have?

d) The most general possible state is

$$|\psi\rangle = c_1 |\updownarrow\rangle + c_2 |\leftrightarrow\rangle$$

where c_1 and c_2 are complex numbers such that $|c_1|^2 + |c_2|^2 = 1$. Suppose that a single photon in this state is subjected to a vertical versus horizontal polarization measurement. Determine the probabilities of the two outcomes.

e) If the state in the previous part was known to be linearly polarized along a line at an angle θ from the vertical, then what are some possibilities for c_1 and c_2?

a) transmitted again $\Rightarrow$ vertical again. State is vertical.

b) Yes (not transmitted $\Rightarrow$ State is horizontal

c) They are associated with two outcomes of one measurement $\Rightarrow$ orthogonal. (and normalized)

d) Prob (horiz) = $|\langle\leftrightarrow|\psi\rangle|^2$ and $\langle\leftrightarrow|\psi\rangle = 0c_1 + 1c_2 = c_2 \Rightarrow$ Prob(horiz) $= |c_2|^2$

Prob (vert) = $|\langle\updownarrow|\psi\rangle|^2$ and $\langle\updownarrow|\psi\rangle = 1c_1 + 0c_2 = c_1 \Rightarrow$ Prob(vert) $= |c_1|^2$

e) Prob (vert) = $\cos^2\theta = |c_1|^2 = \cos^2\theta \Rightarrow c_1 = \cos\theta$

$|c_1|^2 + |c_2|^2 = 1 \Rightarrow \cos^2\theta + |c_2|^2 = 1 \Rightarrow |c_2|^2 = \sin^2\theta \Rightarrow c_2 = \sin\theta$

Thus we can describe linearly polarized light along an angle of θ from the vertical via:

$$\boxed{|\psi\rangle = \cos\theta\,|\updownarrow\rangle + \sin\theta\,|\leftrightarrow\rangle}$$

Then with

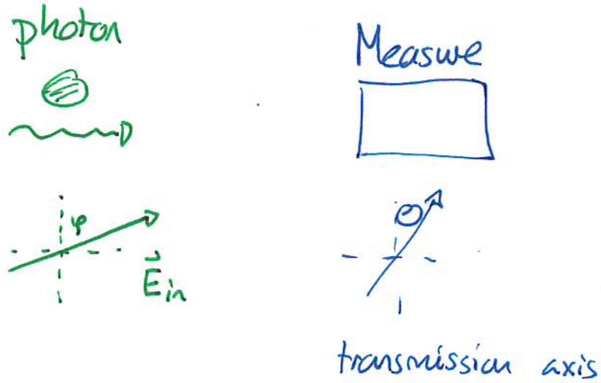
$$|\phi\rangle = a_1\,|\updownarrow\rangle + a_2\,|\leftrightarrow\rangle$$

$$|\psi\rangle = b_1\,|\updownarrow\rangle + b_2\,|\leftrightarrow\rangle$$

the inner product is

$$\langle\phi|\psi\rangle = a_1^* b_1 + a_2^* b_2$$

and we can predict outcomes of measurements, for the following situations



prob transmitted = ??

We can eventually broaden this to circular and elliptical polarization.

Bra vectors

There is a further algebraic automation that is useful for calculating inner products. Currently:

Given
$|\Phi\rangle = a_+|+\hat{z}\rangle + a_-|-\hat{z}\rangle$
$|\Psi\rangle = b_+|+\hat{z}\rangle + b_-|-\hat{z}\rangle$

$\rightarrow$

Extract, by inspection, the coefficients and form inner product
$\langle\Phi|\Psi\rangle = a_+^* b_+ + a_-^* b_-$

This can be done algebraically using raw vectors. Consider the state

$$|\Psi\rangle = c_+|+\hat{z}\rangle + c_-|-\hat{z}\rangle \rightsquigarrow \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

We arbitrarily decided to represent this as a column vector with two complex entries. We could also consider a raw vector with two complex entries. For example

$$(a_+ \;\vdots\; a_-)$$

Such row vectors can be added and multiplied by scalars

1) $(a_+ : a_-) + (b_+ : b_-) = (a_+ + b_+ : a_- + b_-)$

2) $\alpha(a_+ \; a_-) = (\alpha a_+ \; \alpha a_-)$

The set of all such row vectors forms a vector space. This is distinct from the set of all column vectors since we cannot add a raw to a column

$$(a_+ : a_-) + \begin{pmatrix} b_+ \\ b_- \end{pmatrix} = ?? \leftarrow \text{format not even clear.}$$

However we can:

* "act with a row vector on a column vector"
* "operate with a row vector on a column vector"
* "multiply a raw vector with a " "

The key rule is

$$\underbrace{(a_+ \quad a_-)}_{\text{row}} \underbrace{\begin{pmatrix} b_+ \\ b_- \end{pmatrix}}_{\text{column}} = \underbrace{a_+ b_+ + a_- b_-}_{\text{number.}}$$

This can be extended to vectors of larger dimension:

$$\underbrace{(a_1 \quad a_2 \quad a_3 \quad \cdots)}_{\text{row}} \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix}}_{\text{column}} = \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots}_{\text{number.}}$$

row, column → must have same number of entries

In quantum theory the nomenclature is

column vector ⇝ ket $|\cdots\rangle$

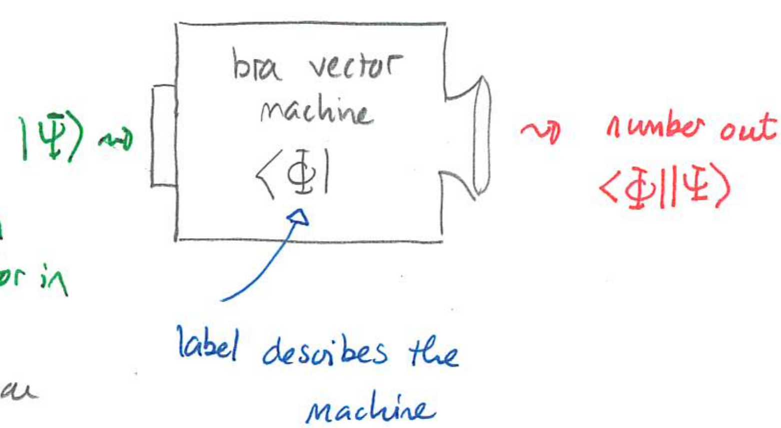
row vector ⇝ bra. $\langle\cdots|$

We can think of any bra vector as a device that operates on a ket vector to produce a number. The effect of bra vector $\langle\Phi|$ on ket $|\Psi\rangle$ is written as

$$\langle\Phi||\Psi\rangle$$

or

$$\langle\Phi|\Psi\rangle$$



and we will see a connection with the inner product soon.

We can associate a bra vector with a ket vector as:

Given ket

$|\Phi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle \rightsquigarrow \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$

Associated bra:

$\langle\Phi| = (a_+^* \;\; a_-^*)$

Given other ket

$|\Psi\rangle = b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle \rightsquigarrow \begin{pmatrix} b_+ \\ b_- \end{pmatrix}$

bra acting on ket

$$\langle\Phi||\Psi\rangle = (a_+^* \;\; a_-^*)\begin{pmatrix} b_+ \\ b_- \end{pmatrix}$$

$$= a_+^* b_+ + a_-^* b_-$$

$$= \langle\Phi|\Psi\rangle$$

inner prod $|\Phi\rangle$ with $|\Psi\rangle$.

So the bra vector construction automates the inner product calculation. Given the basis kets $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ we can construct associated bra vectors:

$$|+\hat{z}\rangle \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \langle+\hat{z}| = (1 \;\; 0)$$

$$|-\hat{z}\rangle \rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \langle-\hat{z}| = (0 \;\; 1)$$

and thus any bra vector can be written as

$$\langle\Phi| = \text{coefficient } \langle+\hat{z}| + \text{coefficient } \langle-\hat{z}|.$$

Then note that

$$\langle+\hat{z}|+\hat{z}\rangle = 1$$

$$\langle+\hat{z}|-\hat{z}\rangle = 0$$

$$\langle-\hat{z}|+\hat{z}\rangle = 0$$

$$\langle-\hat{z}|-\hat{z}\rangle = 0$$