Phys 321 2023

Lecture 4

Thurs: Seminar 12:30 - 1:30 WS

Fri: HW 5pm

Tues: Read text 1.2, 1.3 (bra, matrix)
my notes 38-40

Kets and measurements.

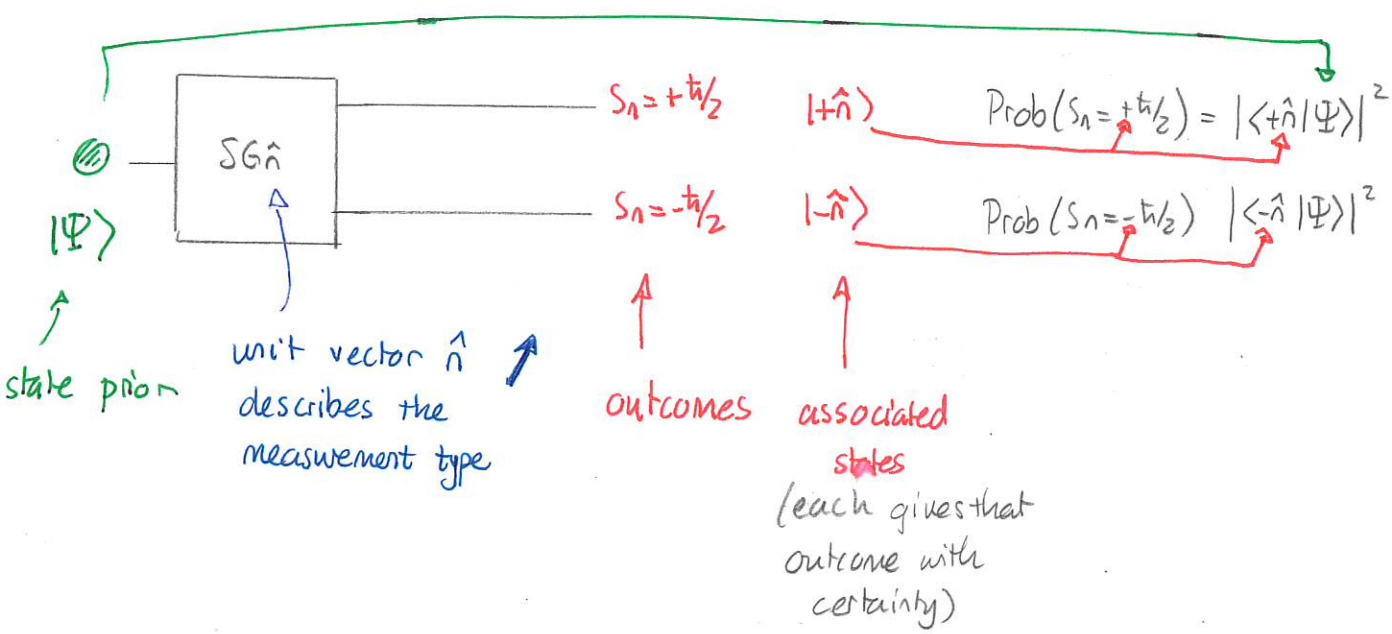
We saw that kets are elements of a vector space that consists of all objects of the form

$$|\Psi\rangle = c_+ |+\hat{z}\rangle + c_- |-\hat{z}\rangle \rightsquigarrow \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

where c_+, c_- are any complex numbers. The inner product of two kets is given via

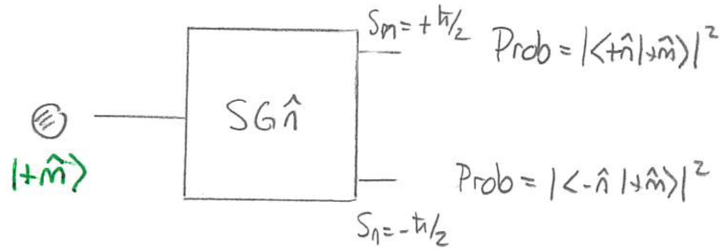
$$\left.\begin{aligned} |\Phi\rangle &= a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle \\ |\Psi\rangle &= b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle \end{aligned}\right\} \Rightarrow \underbrace{\langle\Phi|\Psi\rangle}_{\text{inner prod}} = \underbrace{a_+^* b_+ + a_-^* b_-}_{\text{single complex number}}$$

These fit into a measurement scheme as follows. First we must describe the state prior to measurement $|\Psi\rangle$ and then the measurement type, along with its possible outcomes and associated states



Expressions for $|+\hat{n}\rangle, |-\hat{n}\rangle$ in terms of $|+\hat{z}\rangle, |-\hat{z}\rangle$

In order to analyze the generic illustrated situation we need to be able to calculate inner products such as



$$\langle +\hat{n}|+\hat{m}\rangle \quad , \quad \langle -\hat{n}|+\hat{m}\rangle$$

for any vector directions $\hat{m}, \hat{n}$. Thus we need to express

$$|+\hat{m}\rangle = ??\,|+\hat{z}\rangle + ??\,|-\hat{z}\rangle$$
$$|-\hat{m}\rangle = ??\,|-\hat{z}\rangle + ??\,|-\hat{z}\rangle$$

want these

$$|+\hat{n}\rangle = ??\,|+\hat{z}\rangle + ??\,|-\hat{z}\rangle$$
$$|-\hat{n}\rangle = ??\,|+\hat{z}\rangle + ??\,|-\hat{z}\rangle$$

want these

The components must be such that when they are inserted into the inner product they return the probabilities as described earlier. For example suppose that a particle in state $|+\hat{m}\rangle$ is subjected to $|+\hat{n}\rangle$. Then

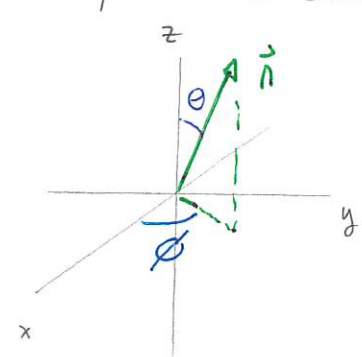
general quantum

$$\text{Prob}\,(S_n = +\hbar/2) = \tfrac{1}{2}(1+\hat{m}\cdot\hat{n}) = |\langle +\hat{n}|+\hat{m}\rangle|^2$$

$$\underbrace{\text{Prob}\,(S_n = -\hbar/2) = \tfrac{1}{2}(1-\hat{m}\cdot\hat{n})}_{\text{previous}} = |\langle -\hat{n}|+\hat{m}\rangle|^2$$

This will require expressing unit vectors in spherical co-ordinates: Then using the usual co-ordinates

$$\hat{n} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$$



1 Unit vector angular coordinates

Determine the angular coordinates for the following unit vectors.

a) $\hat{\mathbf{i}}$

b) $\hat{\mathbf{j}}$

c) $-\hat{\mathbf{i}}$

d) $-\hat{\mathbf{j}}$

e) $\frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}}$

f) $\frac{1}{\sqrt{2}}\hat{\mathbf{i}} - \frac{1}{\sqrt{2}}\hat{\mathbf{j}}$

g) $\frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{k}}$

Answes. a) In the xy plane $\Rightarrow$ $\theta = \pi/2$ $\phi = 0$

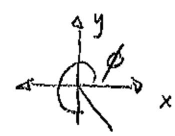
b) " " " " $\Rightarrow$ $\theta = \pi/2$ $\phi = \pi/2$

c) " " " " $\Rightarrow$ $\theta = \pi/2$ $\phi = \pi$

d) " " " " $\Rightarrow$ $\theta = \pi/2$ $\phi = 3\pi/2$

e) " " " " $\Rightarrow$ $\theta = \pi/2$ $\phi = \pi/4$

f) " " " " $\Rightarrow$ $\theta = \pi/2$ $\phi = 7\pi/4$



g) " " xz plane $\Rightarrow$ $\phi = 0$ $\theta = \pi/4$

2 Ket vectors in terms of z basis.

Consider a generic state

$$|+\hat{n}\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$$

where a_+ and a_- are the components in the $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ basis. Suppose that the particle is subjected to a SG $\hat{z}$ measurement. Using the two methods of calculating probabilities show that

$$|a_+| = \cos\left(\frac{\theta}{2}\right)$$
$$|a_-| = \sin\left(\frac{\theta}{2}\right)$$

where θ is the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{z}}$. *Hint: Trigonometric identities are*

$$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \textit{ and}$$
$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right).$$

Answer:

$$\text{Prob}\left(S_z = +\hbar/2\right) = |\langle +\hat{z}|+\hat{n}\rangle|^2$$

$$\| \qquad\qquad \hookrightarrow \quad \langle +\hat{z}|+\hat{n}\rangle = a_+$$

$$\frac{1}{2}(1 + \hat{z}\cdot\hat{n}) \qquad\qquad \Downarrow$$

$$\| \qquad\qquad |\langle +\hat{z}|+\hat{n}\rangle|^2 = |a_+|^2$$

$$\frac{1}{2}(1+\cos\theta)$$

Thus $\quad \frac{1}{2}(1+\cos\theta) = |a_+|^2$

$$\Rightarrow \frac{1}{2}\left(1 + 2\cos^2\left(\frac{\theta}{2}\right) - 1\right) = |a_+|^2 \Rightarrow |a_+|^2 = \cos^2\theta/2$$

$$\Rightarrow |a_+| = \cos\theta/2$$

A similar argument gives

$$|a_-|^2 = \frac{1}{2}(1 - \hat{z}\cdot\hat{n}) = \frac{1}{2}(1-\cos\theta) = \sin^2\theta/2 \Rightarrow |a_-| = \sin\theta/2$$

By considering measurements of S_x and S_y and using similar reasoning, we can arrive at

For any unit vector direction $\hat{n}$ described via spherical co-ordinates θ, ϕ such that

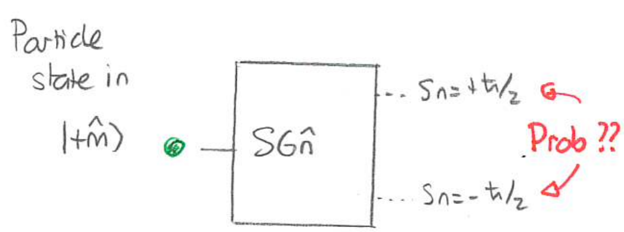
$$\hat{n} = \cos\phi \sin\theta \, \hat{\imath} + \sin\phi \sin\theta \, \hat{\jmath} + \cos\theta \, \hat{k}$$

the kets associated with the S_n measurement are:

$$|+\hat{n}\rangle = \cos(\theta/2) \, |+\hat{z}\rangle + e^{i\phi} \sin(\theta/2) \, |-\hat{z}\rangle$$

$$|-\hat{n}\rangle = \sin(\theta/2) \, |+\hat{z}\rangle - e^{i\phi} \cos(\theta/2) \, |-\hat{z}\rangle$$

This gives the strategy for answering basic state/measurement questions:



Find spherical angles for $\hat{m}$ and write

$$|+\hat{m}\rangle = ??\,|+\hat{z}\rangle + ??\,|-\hat{z}\rangle$$

Find spherical angles for $\hat{n}$ and write

$$|+\hat{n}\rangle = ??\,|+\hat{z}\rangle + ??\,|-\hat{z}\rangle$$

$$|-\hat{n}\rangle = \ldots\,|+\hat{z}\rangle + \ldots\,|-\hat{z}\rangle$$

Use inner product to do calculations

3 Particles in states $\{|+\hat{x}\rangle, |-\hat{x}\rangle\}$ and measurements

Suppose that particles in the states $\{|+\hat{x}\rangle, |-\hat{x}\rangle\}$ are subjected to SG $\hat{y}$ measurements. The aim of this exercise is to use the ket formalism to calculate probabilities of measurement outcomes.

a) Express $|+\hat{x}\rangle$ and $|-\hat{x}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

b) Express $|+\hat{y}\rangle$ and $|-\hat{y}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

c) Suppose that a particle is in state $|+\hat{x}\rangle$. Determine the probabilities of the two SG $\hat{y}$ measurement outcomes.

d) Suppose that a particle is in state $|-\hat{x}\rangle$. Determine the probabilities of the two SG $\hat{y}$ measurement outcomes.

Answer: a) For $\hat{x}$, $\theta = \pi/2$, $\phi = 0$

$$|+\hat{x}\rangle = \cos(\pi/4)|+\hat{z}\rangle + e^{i0}\sin(\pi/4)|-\hat{z}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle = |+\hat{x}\rangle$$

$$|-\hat{x}\rangle = \sin(\pi/4)|+\hat{z}\rangle - e^{i0}\cos(\pi/4)|-\hat{z}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{1}{\sqrt{2}}|-\hat{z}\rangle = |-\hat{x}\rangle$$

b) For $\hat{y}$ $\theta = \pi/2$ $\phi = \pi/2$

$$|+\hat{y}\rangle = \cos(\pi/4)|+\hat{z}\rangle + e^{i\pi/2}\sin(\pi/4)|-\hat{z}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle = |+\hat{y}\rangle$$

$$|-\hat{y}\rangle = \sin(\pi/4)|+\hat{z}\rangle - e^{i\pi/2}\cos(\pi/4)|-\hat{z}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle = |-\hat{y}\rangle$$

c) $\text{Prob}(S_y = +\hbar/2) = |\langle +\hat{y}|+\hat{x}\rangle|^2$

$$\langle +\hat{y}|+\hat{x}\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(\frac{i}{\sqrt{2}}\right)^*\frac{1}{\sqrt{2}} = \frac{1}{2}(1+i) = \frac{1}{2} + i\frac{1}{2}$$

$$|\langle +\hat{y}|+\hat{x}\rangle|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} \Rightarrow \text{Prob}(S_y = +\hbar/2) = 1/2$$

$\text{Prob}(S_y = -\hbar/2) = |\langle -\hat{y}|+\hat{x}\rangle|^2$

$$\langle -\hat{y}|+\hat{x}\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{-i}{\sqrt{2}}\right)^*\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{i}{2}$$

$$|\langle -\hat{y}|+\hat{x}\rangle|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} \Rightarrow \text{Prob}(S_y = -\hbar/2) = 1/2$$

d) $\text{Prob}(S_y = +\hbar/2) = |\langle +y|-\hat{x}\rangle|^2$

$$\langle +\hat{y}|-\hat{x}\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(\frac{i}{\sqrt{2}}\right)^*\left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{2} + i\left(-\frac{1}{2}\right)$$

$$|\langle +\hat{y}|-\hat{x}\rangle|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} \quad \Rightarrow \quad \text{Prob}(S_y = +\hbar/2) = \frac{1}{2}$$

$$\langle -\hat{y}|-\hat{x}\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{2} + i\frac{1}{2} \quad \dots \Rightarrow \text{Prob}(S_y = -\hbar/2) = \frac{1}{2}$$

We have a general rule for expressing any state $|+\hat{n}\rangle$ in the form

$$|+\hat{n}\rangle = \cos\frac{\theta}{2}\,|+\hat{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|-\hat{z}\rangle.$$

Now consider the converse, a generic ket of the form,

$$|\Psi\rangle = c_+\,|+\hat{z}\rangle + c_-\,|-\hat{z}\rangle$$

where $|c_+|^2 + |c_-|^2 = 1$. A mathematical theorem states:

Given c_+, c_- such that $|c_+|^2+|c_-|^2=1$ and

$$|\Psi\rangle = c_+|+\hat{z}\rangle + c_-|-\hat{z}\rangle$$

there exist $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ and $0 \le \alpha \le 2\pi$ such that

$$c_+ = e^{i\alpha}\cos(\theta/2)$$

$$c_- = e^{i\alpha}e^{i\phi}\sin(\theta/2)$$

and

$$|\Psi\rangle = e^{i\alpha}\left\{\cos(\theta/2)\,|+\hat{z}\rangle + e^{i\phi}\sin(\theta/2)\,|-\hat{z}\rangle\right\}.$$

It is a straightforward exercise to show that the factor $e^{i\alpha}$, called a global phase, does not affect any probabilities. Thus.

For every normalized state $|\Psi\rangle$ of a spin-$\frac{1}{2}$ particle there exists some direction $\hat{n}$ such that

$$|\Psi\rangle = |+\hat{n}\rangle$$

Thus the totality of all (pure) spin-$\frac{1}{2}$ particle states is

$$\{|+\hat{n}\rangle : \text{all unit vectors } \hat{n}\}$$

Additionally it follows that.

Given a state $|\Psi\rangle$

Can find θ, ϕ, α

$|\Psi\rangle = e^{i\alpha} |+\hat{n}\rangle$

where $\hat{n}$ is the unit vector corresponding to angles θ, ϕ

Measure S_n and for this state get

$S_n = +\hbar/2$

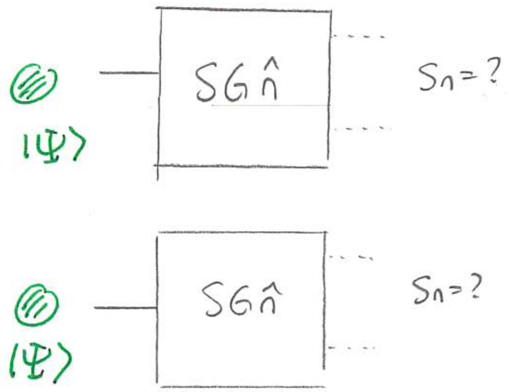
with certainty

So

For every state $|\Psi\rangle$, there exists some measurement such that this measurement will yield one outcome with certainty.

Mean and uncertainty

Suppose that we have a collection of particles all in the same state and we perform the same measurement on each. We could then calculate



the sample average of these

$$\bar{S}_n = \frac{+\frac{\hbar}{2}(\text{number}+) + \left(-\frac{\hbar}{2}\right)(\text{number}-)}{\text{total number particles}}$$

This in fact occurs in many experimental situations. Quantum theory can predict statistically what the sample average should be. It does so via the mean or expectation value. For the illustrated measurement the mean is

$$\langle S_n \rangle = \left(+\frac{\hbar}{2}\right) \times \text{Prob}(S_n = +\hbar/2) + \left(-\frac{\hbar}{2}\right) \times \text{Prob}(S_n = -\hbar/2)$$

$$= +\frac{\hbar}{2} |\langle +\hat{n}|\Psi\rangle|^2 - \frac{\hbar}{2} |\langle -\hat{n}|\Psi\rangle|^2$$

$$= +\frac{\hbar}{2} \left\{ |\langle +\hat{n}|\Psi\rangle|^2 - |\langle -\hat{n}|\Psi\rangle|^2 \right\}$$

The mean is an idealization and the sample average will fluctuate around it. The fluctuations can be quantified by the standard deviation or uncertainty

$$\Delta S_n = \text{average}\left(S_n - \langle S_n \rangle \right)^2$$

Standard algebra gives

$$\Delta S_n = \sqrt{\langle S_n^2 \rangle - \langle S_n \rangle^2}$$

where

$$\langle S_n^2 \rangle = \left(+\tfrac{\hbar}{2}\right)^2 \text{Prob}\left(S_n = +\hbar/2\right) + \left(-\tfrac{\hbar}{2}\right)^2 \text{Prob}\left(S_n = -\hbar/2\right)$$

$$= \frac{\hbar^2}{4} \underbrace{\left[\text{Prob}\left(S_n = +\hbar/2\right) + \text{Prob}\left(S_n = -\hbar/2\right)\right]}_{=1}$$

$$= \frac{\hbar^2}{4}.$$

These are examples of a more general framework. Let M be some measurement.

List

Outcomes	Probabilities
m_1	p_1
m_2	p_2
m_3	p_3
$\vdots$	$\vdots$

Mean (expectation value)

$$\langle m \rangle = \sum_{\text{all outcomes}} m_j p_j$$

Standard deviation (uncertainty)

$$\Delta m = \sqrt{\langle m^2 \rangle - \langle m \rangle^2}$$

where

$$\langle m^2 \rangle = \sum_{\text{all outcomes}} m_j^2 p_j$$