Phys 321 2023

Lecture 3

Tues: HW by 5pm

Thurs: Seminar 12:30 - 1:30pm WS

Text: 2.2 My notes pg 32 - 38

Fri: HW by 5pm

Note: Homework solutions posted to D2L

Vector description of states

The set of all possible (pure) states for a spin-1/2 system are:

$|+\hat{n}\rangle$ where $\hat{n}$ is any unit vector direction in three dimensional space

We aim to provide a mathematical structure for such pure states that will enable:

1) algebraic operations using kets
2) calculation of probabilities of measurement outcomes and subsequent states using kets
3) mathematical representation of evolution processes using kets.

It will emerge that a suitable mathematical framework is one where

Every ket is an element of one particular complex vector space.

We illustrate this using the special pair of kets $|+\hat{z}\rangle$ and $|-\hat{z}\rangle$ and make the correspondence

$$|+\hat{z}\rangle \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-\hat{z}\rangle \rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This allows us to form an arbitrary superposition:

$$c_+ |+\hat{z}\rangle + c_- |-\hat{z}\rangle \rightsquigarrow c_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_- \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_+ \\ c_- \end{pmatrix}.$$

Thus we can form a general ket

$$\boxed{|\Psi\rangle = c_+ |+\hat{z}\rangle + c_- |-\hat{z}\rangle \leftrightsquigarrow \begin{pmatrix} c_+ \\ c_- \end{pmatrix}}$$

ket label

complex numbers

representation of ket

We can use the representation to guide how formal algebraic manipulations unfold. So consider two states

$$|\Psi_1\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$$

$$|\Psi_2\rangle = b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle$$

Column vector representation

$$|\Psi_1\rangle \leftrightsquigarrow \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \quad \Psi_2 \leftrightsquigarrow \begin{pmatrix} b_+ \\ b_- \end{pmatrix}$$

$$|\Psi_1\rangle + |\Psi_2\rangle \leftrightsquigarrow \begin{pmatrix} a_+ \\ a_- \end{pmatrix} + \begin{pmatrix} b_+ \\ b_- \end{pmatrix}$$

$$= \begin{pmatrix} a_+ + b_+ \\ a_- + b_- \end{pmatrix}$$

$$\Updownarrow$$

$$(a_+ + b_+) |+\hat{z}\rangle + (a_- + b_-) |-\hat{z}\rangle$$

Want

$$|\Psi_1\rangle + |\Psi_2\rangle = ?? \, |+\hat{z}\rangle + ?? \, |-\hat{z}\rangle$$

$$= (a_+ + b_+) |+\hat{z}\rangle + (a_- + b_-) |-\hat{z}\rangle$$

A similar rule applies for scalar multiplication: Thus:

If
$$|\Psi_1\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$$
$$|\Psi_2\rangle = b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle$$

then

$$|\Psi_1\rangle + |\Psi_2\rangle = (a_+ + b_+)|+\hat{z}\rangle + (a_- + b_-)|-\hat{z}\rangle$$

and
$$\alpha|\Psi_1\rangle = \alpha a_+ |+\hat{z}\rangle + \alpha a_- |-\hat{z}\rangle.$$

With these definitions we can see that:

1) the set of all objects
$$|\Psi\rangle = c_+ |+\hat{z}\rangle + c_- |-\hat{z}\rangle$$
where c_+, c_- are complex numbers satisfies the requirements of a vector space over complex numbers. That is that any two of these can be added to produce a third and any can be multiplied by a complex scalar to produce another.

2) note that we cannot manipulate the labels when adding. So
$$|+\hat{z}\rangle + |-\hat{z}\rangle = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \neq |+\hat{z} + (-\hat{z})\rangle = |0\rangle$$

3) we still need rules for manipulating $|+\hat{x}\rangle + |+\hat{z}\rangle$ and dealing with different labels

Inner product

Recall that ordinary vectors in two dimensional space can be equipped with a dot product. So

If $\vec{u} = u_x \hat{i} + u_y \hat{j}$ then $\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y$

$\vec{v} = v_x \hat{i} + v_y \hat{j}$

This can be generalized to vectors in any dimensional and to vectors with complex coefficients. The generalization is called the inner product.

Let $|\Phi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$

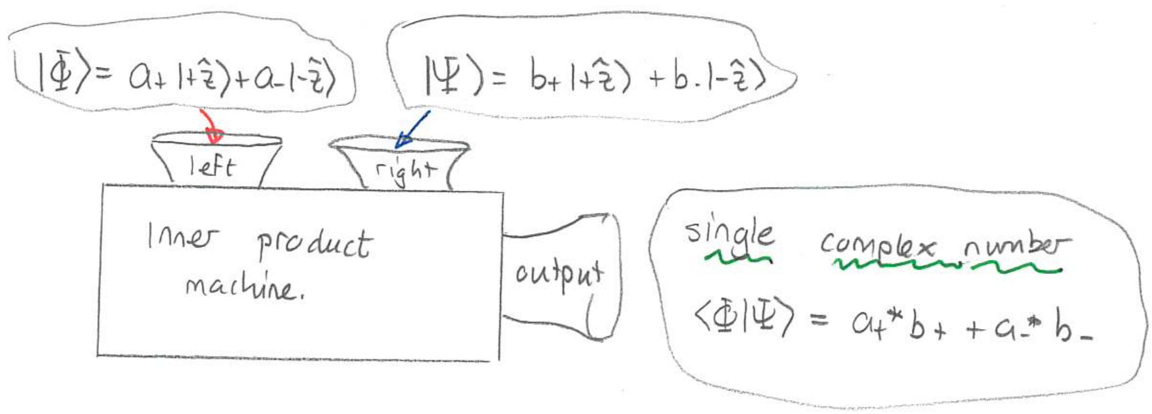
$|\Psi\rangle = b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle$

then the inner product of $|\Phi\rangle$ with $|\Psi\rangle$ is denoted $\langle\Phi|\Psi\rangle$ and is defined as

$$\langle\Phi|\Psi\rangle := a_+^* b_+ + a_-^* b_-$$

left right left right left right

We can view the inner product as a machine.



1 Ket algebra

Consider the kets

$$|\Phi\rangle = 2\,|{+\hat{z}}\rangle + 2i\,|{-\hat{z}}\rangle$$
$$|\Psi\rangle = 3i\,|{+\hat{z}}\rangle + 2\,|{-\hat{z}}\rangle$$

a) Determine an expression for $|\Phi\rangle + |\Psi\rangle$ using both the ket representation and also the column vector representation.

b) Calculate $\langle\Phi|\Psi\rangle$.

c) Calculate $\langle\Psi|\Phi\rangle$. How is this related to $\langle\Phi|\Psi\rangle$?

d) Calculate $\langle\Phi|\Phi\rangle$.

Answer: a) ket representation

$$|\Phi\rangle + |\Psi\rangle = \left[2|{+\hat{z}}\rangle + 2i|{-\hat{z}}\rangle\right] + \left[3i|{+\hat{z}}\rangle + 2|{-\hat{z}}\rangle\right]$$

$$= (2+3i)|{+\hat{z}}\rangle + (2i+2)|{-\hat{z}}\rangle$$

column vector representation

$$|\Phi\rangle \to \begin{pmatrix} 2 \\ 2i \end{pmatrix}$$
$$|\Psi\rangle \to \begin{pmatrix} 3i \\ 2 \end{pmatrix}$$

add $\begin{pmatrix} 2+3i \\ 2i+2 \end{pmatrix} \to (2+3i)|{+\hat{z}}\rangle + (2i+2)|{-\hat{z}}\rangle$

b) $\langle\Phi|\Psi\rangle = 2^* \cdot 3i + (2i)^*(2) = 6i - 4i = 2i$

c) $\langle\Psi|\Phi\rangle = -3i \cdot 2 + 2 \cdot 2i = -6i + 4i = -2i$

$\langle\Psi|\Phi\rangle = (\langle\Phi|\Psi\rangle)^*$

d) $\langle\Phi|\Phi\rangle = 2^* 2 + (2i)^*(2i) = 4 + (-2i)(2i)$

$= 4 + 4$

$= 8$

The definition of the inner product implies:

1) If $|\Psi\rangle = \alpha_1 |\Psi_1\rangle + \alpha_2 |\Psi_2\rangle$ then

$$\langle \Phi | \Psi \rangle = \alpha_1 \langle \Phi | \Psi_1 \rangle + \alpha_2 \langle \Phi | \Psi_2 \rangle$$

2) if $|\Phi\rangle = \beta_1 |\Phi_1\rangle + \beta_2 |\Phi_2\rangle$ then

$$\langle \Phi | \Psi \rangle = \beta_1^* \langle \Phi_1 | \Psi \rangle + \beta_2^* \langle \Phi_2 | \Psi \rangle.$$

3) $\langle \Psi | \Phi \rangle = \left(\langle \Phi | \Psi \rangle \right)^*$

4) $\langle \Psi | \Psi \rangle$ is real.

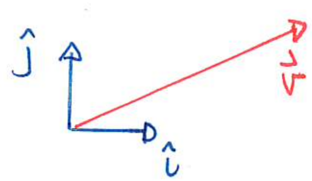
Two additional pieces of terminology are

A ket $|\Psi\rangle$ is normalized $\Leftrightarrow$ $\langle \Psi | \Psi \rangle = 1$

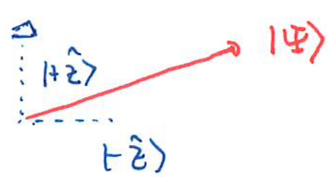
and

Two kets $|\Phi\rangle, |\Psi\rangle$ are orthogonal $\Leftrightarrow$ $\langle \Phi | \Psi \rangle = 0$

The a set of kets is called orthonormal if they are each normalized and any pair of different kets is orthogonal. We can use the definitions of the inner product to show that $\{ |+\hat{z}\rangle, |-\hat{z}\rangle \}$ are orthonormal. We can think of these as analogous to the conventional vectors $\hat{\imath}, \hat{\jmath}$



$\approx$



$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = 1$

$\hat{\imath} \cdot \hat{\jmath} = 0$

any vector $\vec{v}$:

$\vec{v} = v_x \hat{\imath} + v_y \hat{\jmath}$

real

$\langle +\hat{z} | +\hat{z} \rangle = \langle -\hat{z} | -\hat{z} \rangle = 1$

$\langle +\hat{z} | -\hat{z} \rangle = 0$

any ket

basis vectors/kets

$|\Psi\rangle = c_+ |+\hat{z}\rangle + c_- |-\hat{z}\rangle$

complex

Note the special convenient results:

$$\langle +\hat{z}|+\hat{z}\rangle = 1$$
$$\langle -\hat{z}|-\hat{z}\rangle = 1$$
$$\langle +\hat{z}|-\hat{z}\rangle = 0$$
$$\langle -\hat{z}|+\hat{z}\rangle = 0$$

So we can now construct a set of kets:

$$\left\{ |\Psi\rangle = c_+|+\hat{z}\rangle + c_-|-\hat{z}\rangle \quad \text{complex numbers } c_+, c_- \right\}.$$

and these form a vector space with $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ as an orthonormal basis. We shall soon:

1) describe a physical meaning to an orthonormal basis and show that there are many possible orthonormal bases.
2) show how to use the inner product to determine probabilities of measurement outcomes.

3) show how to represent states in terms of $|+\hat{z}\rangle, |-\hat{z}\rangle$. Thus

$$|+\hat{n}\rangle = ??\,|+\hat{z}\rangle + ??\,|-\hat{z}\rangle$$

2 Ket orthonormality

a) Consider the kets

$$|\Phi\rangle = |+\hat{z}\rangle \text{ and}$$
$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,|+\hat{z}\rangle - \frac{1}{\sqrt{2}}\,|-\hat{z}\rangle .$$

Check whether these are orthonormal.

b) Consider the kets

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\,|+\hat{z}\rangle + \frac{i}{\sqrt{2}}\,|-\hat{z}\rangle \text{ and}$$
$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,|+\hat{z}\rangle - \frac{i}{\sqrt{2}}\,|-\hat{z}\rangle .$$

Check whether these are orthonormal.

Answer: a) $|\Phi\rangle = 1\,|+\hat{z}\rangle + 0\,|-\hat{z}\rangle$

$\Rightarrow \langle\Phi|\Phi\rangle = 1\,1 + 0\cdot 0 = 1$

$\langle\Psi|\Psi\rangle = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$

These are normal. Check for orthogonal.

$\langle\Phi|\Psi\rangle = 1\cdot\left(\frac{1}{\sqrt{2}}\right) + 0\cdot\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \neq 0$

They are not orthogonal

b) $\langle\Phi|\Phi\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(\frac{i}{\sqrt{2}}\right)^* \left(\frac{i}{\sqrt{2}}\right) = \frac{1}{2} + \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$

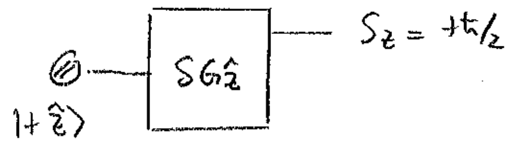
This is normal

$\langle\Phi|\Psi\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(\frac{i}{\sqrt{2}}\right)^* \left(\frac{-i}{\sqrt{2}}\right) = \frac{1}{2} + \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{2} = 0$

These are orthonormal

Orthonormal kets and measurements.

Consider the two orthonormal kets $|+\hat{z}\rangle, |-\hat{z}\rangle$. These are related to measurements of S_z. We have



and

Each of these kets gives one outcome with certainty. The two outcomes are mutually exclusive $(S_z = +\hbar/2) \neq (S_z = -\hbar/2)$. In this sense the two kets $|+\hat{z}\rangle, |-\hat{z}\rangle$ are each associated with one outcome of a single measurement (S_z). A central tenet of quantum theory is:

> Two kets associated with mutually exclusive outcomes of a single measurement are orthogonal.

Thus we have

1) measurement $S_x \rightsquigarrow$ associated kets $|+\hat{x}\rangle, |-\hat{x}\rangle$

$$\Rightarrow \{|+\hat{x}\rangle, |-\hat{x}\rangle\} \text{ are orthogonal.}$$

2) measurement $S_n \rightsquigarrow$ $\{|+\hat{n}\rangle, |-\hat{n}\rangle\}$ are orthogonal.

The converse is also a tenet

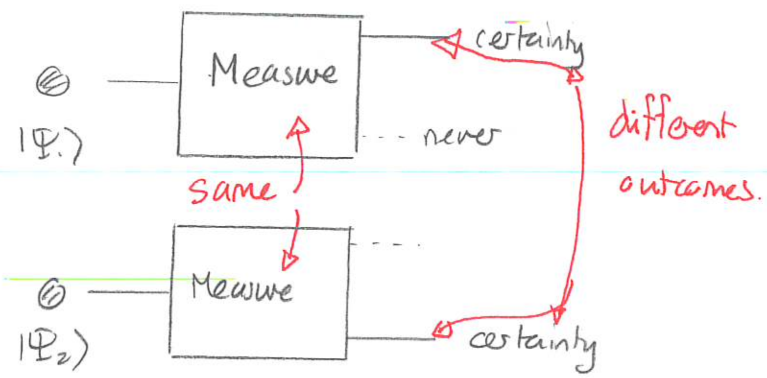
> Two orthogonal kets are associated with mutually exclusive outcomes of one measurement.

Example: The kets

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

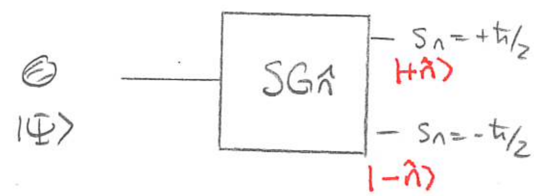
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

are orthogonal. So there exists one measurement



Kets and probabilities

Suppose that we subject a particle in a known state to a measurement. How can we determine the probabilities of the outcomes. The procedure is:



Identify the states associated with the two measurement outcomes (i.e. the states that give these with certainty) $\rightarrow$ e.g. for S_n measurement the states are $|+\hat{n}\rangle$, $|-\hat{n}\rangle$ ($S_n = +\hbar/2$, $S_n = -\hbar/2$)

$\downarrow$

Identify the state of the particle prior to measurement $\rightarrow$ $|\Psi\rangle$

Then for spin-1/2 particles

$$\text{Prob}(S_n = +\hbar/2) = |\langle +\hat{n}|\Psi\rangle|^2$$

state associated with $S_n = +\hbar/2$

$$\text{Prob}(S_n = -\hbar/2) = |\langle -\hat{n}|\Psi\rangle|^2$$

state associated with $S_n = -\hbar/2$

3 Probabilities of measurement outcomes

The most general ket is

$$|\Psi\rangle = c_+ |+\hat{z}\rangle + c_- |-\hat{z}\rangle$$

where c_+ and c_- are two complex numbers. Determine an expression for the probabilities with which each of the outcomes for a SG $\hat{z}$ measurement will occur.

Answer:

$$\text{Prob}(S_z = +\hbar/2) = |\langle +\hat{z}|\Psi\rangle|^2$$

$$\text{Prob}(S_z = -\hbar/2) = |\langle -\hat{z}|\Psi\rangle|^2$$

Now

$$\langle +\hat{z}|\Psi\rangle = 1c_+ + 0c_- = c_+$$

$$\langle -\hat{z}|\Psi\rangle = 0c_+ + 1c_- = c_-$$

$$\Rightarrow \quad \text{Prob}(S_z = +\hbar/2) = |c_+|^2$$

$$\text{Prob}(S_z = -\hbar/2) = |c_-|^2$$

Note that we require that the two probabilities add to 1. Thus

$|c_+|^2 + |c_-|^2 = 1 \quad \Rightarrow \quad \langle\Psi|\Psi\rangle = 1$.

Thus

Any ket (state) for a quantum system, $|\Psi\rangle$, must be normalized

$$\langle\Psi|\Psi\rangle = 1$$