Fri: HW1 by 5pm

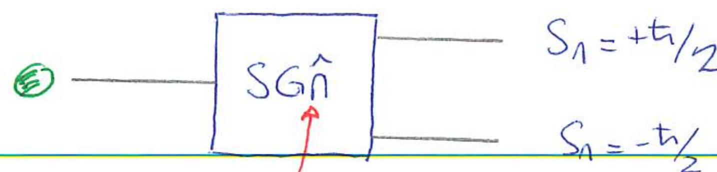
Tues: HW2 by 5pm

Spin-1/2 Particles

Recall that a spin-1/2 particle can be characterized by outcomes of measurements of a spin component

Possible outcomes (only two)

Prepare particle in known state



Choose component to measure
$\Rightarrow$ specify unit vector direction $\hat{n}$ of component

The crucial question is

"Given a known state for the particle prior to measurement what will the outcome of the measurement be?"

A general observation is that we cannot predict measurement outcomes with certainty. However, quantum theory allows us to predict the probabilities with which the outcomes occur

outcomes

quantum theory gives

known state

Prob $(S_n = +\hbar/2)$

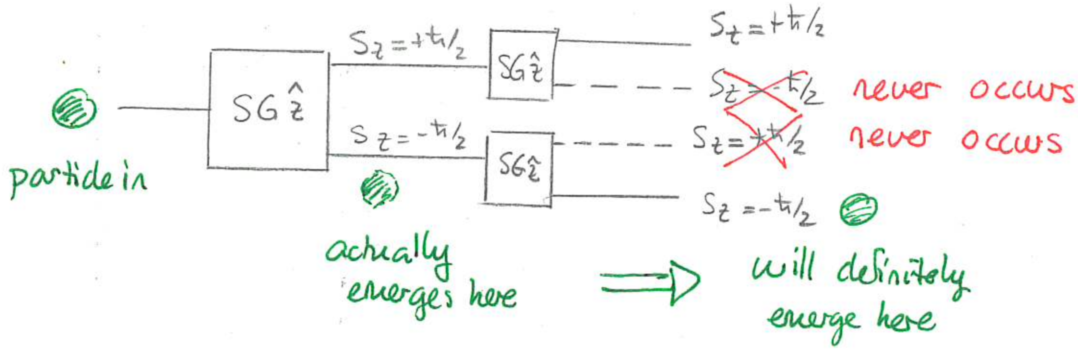
Prob $(S_n = -\hbar/2)$

(what math) + (what math) $\xrightarrow{\text{gives?}}$

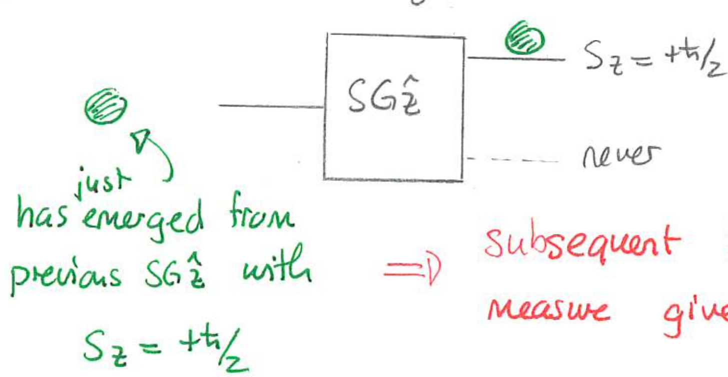
We can determine mathematical rules by considering successive measurements.

Successive measurements (same component)

Suppose that we follow one measurement by another of the same type. For example we could perform successive S_z measurements on the same particle.



This is observed in analogous experiments involving photons. We can then see that there are states of the particle which produce particular outcomes with certainty



$\Longrightarrow$ Subsequent SG$\hat{z}$ measure gives $S_z = +\hbar/2$ WITH CERTAINTY

We will use the "ket" notation to represent such states. This is

$$|\text{label}\rangle$$

label $\rightarrow$ describes state

In the above situation

$$\underbrace{|+\hat{z}\rangle}_{\text{state of particle}} \Longrightarrow \text{If one measures } S_z \text{ then one obtains } S_z = +\hbar/2 \text{ with certainty.}$$

Similarly

$\underbrace{|-\hat{z}\rangle}_{\text{state}} \Rightarrow$ If one measures S_z one obtains $S_z = -\hbar/2$ with certainty

We can list states for all possible components / directions.

> Let $\hat{n}$ be any unit vector direction in three dimensional space.
> Then the following states exist:
>
> $|+\hat{n}\rangle \Rightarrow$ measure $S_n \Rightarrow$ get $S_n = +\hbar/2$ with certainty
>
> $|-\hat{n}\rangle \Rightarrow$ " $S_n \Rightarrow$ get $S_n = -\hbar/2$ with certainty

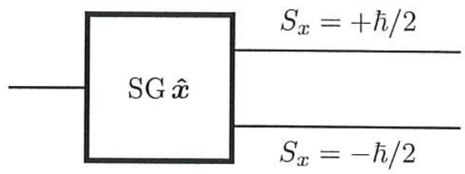
This uses the fact that:

If an SG$\hat{n}$ measurement yields $S_n = +\hbar/2$ then another SG$\hat{n}$ measurement immediately after yields $S_n = +\hbar/2$ with certainty

If an SG$\hat{n}$ measurement yields $S_n = -\hbar/2$ then another SG$\hat{n}$ measurement immediately after yields $S_n = -\hbar/2$ with certainty.

1 SG $\hat{x}$ measurements

Various spin-1/2 particles are all prepared in exactly the same state and are subjected to the indicated Stern-Gerlach measurement.



a) Consider a particle initially in the state $|+\hat{x}\rangle$. What could the measurement outcome be?

b) Consider a particle initially in the state $|-\hat{x}\rangle$. What could the measurement outcome be?

c) Suppose that four particles are all prepared in the same state. They are sent, one at a time, through the measuring device. Three measurement outcomes are $S_x = +\hbar/2$ and one is $S_x = -\hbar/2$. Can we say that the state (of all four) prior to measurement was $|+\hat{x}\rangle$ or that it was $|-\hat{x}\rangle$?

d) Suppose that a single particle yields outcome $S_x = +\hbar/2$. Can one say that the state of the particle prior to measurement is $|+\hat{x}\rangle$ with certainty? What is the state of this particle after measurement?

e) Suppose that a measurement yields $S_x = -\hbar/2$. What is the state of the particle after the measurement?

f) How could this be used to prepare particles in the state $|-\hat{x}\rangle$?

g) Consider a particle initially in the state $|+\hat{z}\rangle$. What could the measurement outcome be?

Answer: a) It must be $S_x = +\hbar/2$ with certainty.

b) It must be $S_x = -\hbar/2$ " "

c) If they were all $|+\hat{x}\rangle$ then all outcomes would be $S_x = +\hbar/2$
" " " " $|-\hat{x}\rangle$ " " " " " $S_x = -\hbar/2$

So they cannot all be $|+\hat{x}\rangle$ nor all $|-\hat{x}\rangle$

d) We cannot say before - see part c) for a counterexample.
A subsequent S_x measurement gives $S_x = +\hbar/2$ with certainty so after the illustrated measurement state is $|+\hat{x}\rangle$

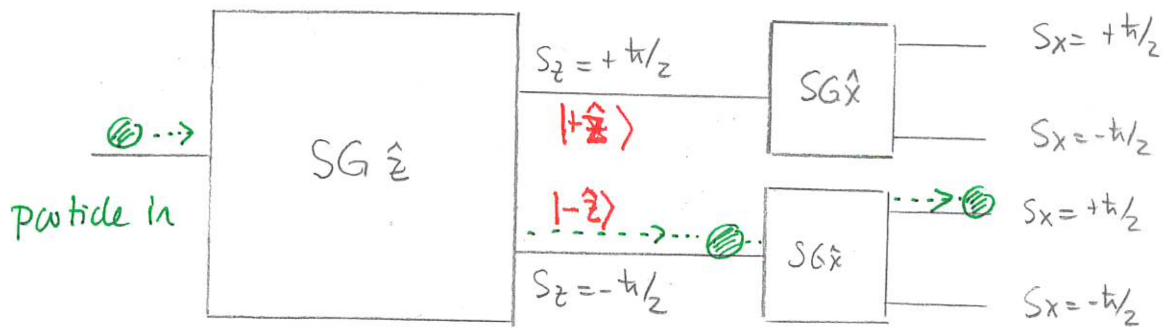
e) $|-\hat{x}\rangle$ because another S_x gives $S_x = -\hbar/2$ with certainty.

f) Measure S_x - keep particles for which measurement gave $S_x = -\hbar/2$.

g) Cannot say at this point.

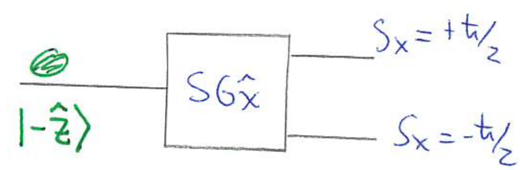
Successive measurements: different types

The more general and more interesting situations are where the successive measurements are of different types. In the case of spin-½ particles this means measuring different components of spin.



First measure S_z Second measure S_x

The diagram illustrates a particular pair of outcomes $\left(S_z = -\hbar/2, S_x = +\hbar/2\right)$. In general all four combinations are possible unlike the situation where the same measurement is repeated (only two possible combinations possible). Consider the particular trajectory. We known that after the SG $\hat{z}$ measurement, the state is $|-\hat{z}\rangle$. So we can focus on the illustrated situation. We would like to determine the probability with which the two possible outcomes occur. Denote



$\text{Prob}(S_x = +\hbar/2) \equiv$ probability that measuring S_x gives $S_x = +\hbar/2$

$\text{Prob}(S_x = -\hbar/2) \equiv$ " " " S_x " $S_x = -\hbar/2$

The rule for calculating these must be inferred from experimental evidence.

The rule for spin-½ particles is:

Prepare using direction $\hat{m}$

Measure along $\hat{n}$

POSSIBLE outcomes

Then

Particle in state $|+\hat{m}\rangle$ into $SG\hat{n}$ $\rightsquigarrow$ $\begin{cases} S_n = +\hbar/2 & \text{probability } \frac{1}{2}(1+\hat{m}\cdot\hat{n}) \\ S_n = -\hbar/2 & \text{probability } \frac{1}{2}(1-\hat{m}\cdot\hat{n}) \end{cases}$

Particle in state $|-\hat{m}\rangle$ into $SG\hat{n}$ $\rightsquigarrow$ $\begin{cases} S_n = +\hbar/2 & \text{probability } \frac{1}{2}(1-\hat{m}\cdot\hat{n}) \\ S_n = -\hbar/2 & \text{probability } \frac{1}{2}(1+\hat{m}\cdot\hat{n}) \end{cases}$

2 Successive measurements

a) A spin-1/2 particle is subjected to a SG $\hat{\boldsymbol{y}}$ device and the measurement outcome is $S_y = +\hbar/2$. This particle is then subjected to a SG $\hat{\boldsymbol{z}}$ measurement. Determine the probabilities of the two measurement outcomes.

b) A spin-1/2 particle is subjected to a SG $\hat{\boldsymbol{y}}$ device and the measurement outcome is $S_y = -\hbar/2$. This particle is then subjected to a SG $\hat{\boldsymbol{z}}$ measurement. Determine the probabilities of the two measurement outcomes.

c) A spin-1/2 particle in the state $|-\hat{\boldsymbol{z}}\rangle$ is subjected to a measurement of S_n where $\hat{\mathbf{n}}$ lies in the xz plane midway between the two positive axes. Determine the probabilities of the two measurement outcomes.

Answer: a) Emerges with $S_y = +\hbar/2 \Rightarrow$ state is $|+\hat{y}\rangle$. So

$|+\hat{y}\rangle$ into $SG\hat{z}$ $\rightsquigarrow$ $S_z = +\hbar/2$ prob $\frac{1}{2}(1+\hat{y}\cdot\hat{z}) = \frac{1}{2}$

$S_z = -\hbar/2$ prob $\frac{1}{2}(1-\hat{y}\cdot\hat{z}) = \frac{1}{2}$

b) $|-\hat{y}\rangle$ into $SG\hat{z}$ $\rightsquigarrow$ $S_z = +\hbar/2$ prob $\frac{1}{2}(1-\hat{y}\cdot\hat{z}) = \frac{1}{2}$

$S_z = -\hbar/2$ prob $\frac{1}{2}(1+\hat{y}\cdot\hat{z}) = \frac{1}{2}$

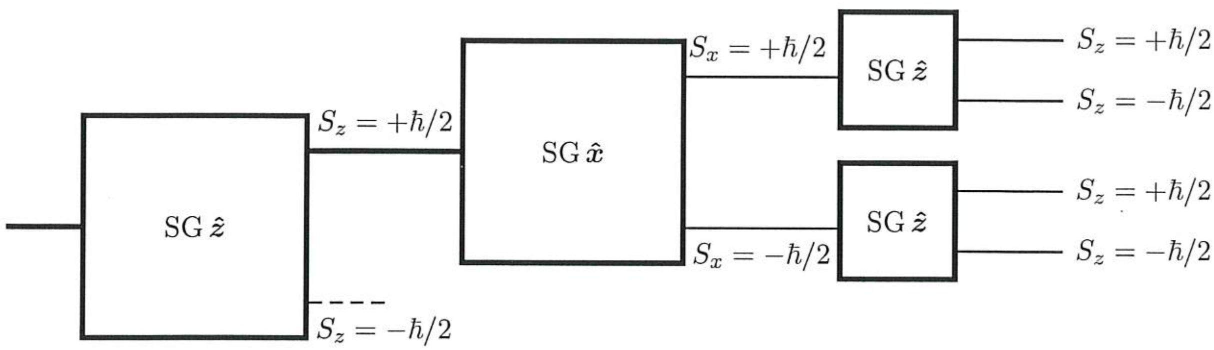
c) $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{z})$

$$\text{Prob}(S_n = +\hbar/2) = \frac{1}{2}(1-\hat{z}\cdot\hat{n}) = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$$

$$\text{Prob}(S_n = -\hbar/2) = \frac{1}{2}(1+\hat{z}\cdot\hat{n}) = \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)$$

3 Three measurements

Spin-1/2 particles are subjected a series of three SG measurements as illustrated. The particles that emerge from the first device with $S_z = -\hbar/2$ are immediately removed.



a) Suppose that the particle emerges from the SG $\hat{x}$ measurement with $S_x = +\hbar/2$. Show that the probability with which the particle emerges from the subsequent SG $\hat{z}$ measurement with $S_z = +\hbar/2$ is 1/2. Repeat this for a particle that emerges with $S_z = -\hbar/2$ is 1/2. Repeat this entire analysis for the case where the particle emerges from the SG $\hat{x}$ measurement with $S_x = -\hbar/2$.

b) What are the probabilities with which any particle will emerge in any of the four outputs on the right, given that it does emerge with $S_z = +\hbar/2$ from the first device.

c) In the illustrated situation what is the state of the particle that emerges after the first measurement? Could this be the state after it emerges from the second measurement?

Answer: a) It emerges in state $|+\hat{x}\rangle$. Prob $S_z = +\hbar/2 = \frac{1}{2}(1 + \hat{x}\cdot\hat{z}) = \frac{1}{2}$

Prob $S_z = -\hbar/2 = \frac{1}{2}(1 - \hat{x}\cdot\hat{z}) = \frac{1}{2}$

If $S_x = -\hbar/2$ state is $|-\hat{x}\rangle$ Prob $S_z = +\hbar/2 = \frac{1}{2}(1 - \hat{x}\cdot\hat{z}) = \frac{1}{2}$

Prob $S_z = -\hbar/2 = \frac{1}{2}(1 + \hat{x}\cdot\hat{z}) = \frac{1}{2}$.

b) They will all be of the form $\text{Prob}(S_x = +\hbar/2)\,\text{Prob}(S_z = +\hbar/2) = \frac{1}{4}$

$\Rightarrow$ all $1/4$.

c) $|+\hat{z}\rangle$ after first. Cannot be this after second $\Rightarrow$ sometimes get $S_z = -\hbar/2$.

Possible states for spin-½ particles

We have seen various possible states for spin-½ particles: $|+\hat{z}\rangle, |-\hat{z}\rangle, |+\hat{x}\rangle, \ldots$ Then we can form an infinite collection of states

For any unit vector direction $\hat{n}$ $\rightsquigarrow$ There are two states $|+\hat{n}\rangle, |-\hat{n}\rangle$

These are the only states that can yield measurement outcomes with certainty (there is an extension to states for which there is no spin measurement that yields one outcome with certainty).

Noting that $|-\hat{n}\rangle$ is the same as $|+(-\hat{n})\rangle$ for the unit vector $-\hat{n}$ we get.

> The set of all states that yield an outcome of a spin measurement with certainty is:
>
> $$\{ |+\hat{n}\rangle : \text{all possible unit vectors } \hat{n} \}$$

We now develop a mathematical structure for these states / kets.

States as vectors

Consider the two basic states $|+\hat{z}\rangle, |-\hat{z}\rangle$. Can we attach some meaning to algebraic combinations of these? An example would be

$$\frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle.$$

One way would be to think of the two kets as column vectors. So we could associate:

$$\boxed{\begin{aligned} |+\hat{z}\rangle &\longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix} \\ |-\hat{z}\rangle &\longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}\end{aligned}}$$

Then

$$\frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1/\sqrt{2}\\0\end{pmatrix} + \begin{pmatrix}0\\1/\sqrt{2}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

This would give a mathematical meaning to any combination. Thus

$$\boxed{\begin{gathered}\text{For any complex numbers } c_+, c_- \\ c_+|+\hat{z}\rangle + c_-|-\hat{z}\rangle \longleftrightarrow \begin{pmatrix}c_+\\c_-\end{pmatrix}\end{gathered}}$$

By translating between kets and column vectors we can, in effect, do mathematical manipulations of kets.

This will be a key idea in the mathematics for quantum systems

The states of a quantum system correspond to (column) vectors with complex number entries.