Quantum Theory I: Homework 24

Due: 9 May 2023

1 Angular position measurements

Consider a particle in a central potential and let $|l, m\rangle$ denote the conventional angular momentum states. An ensemble consists of particles, each prepared in the same state. The angular positions are measured by a device that can determine whether the particle is in any of the following regions

$$\begin{aligned}
\text{Region I:} \quad & 0 \leqslant \theta \leqslant \pi/3 \\
\text{Region II:} \quad & \pi/3 \leqslant \theta \leqslant 2\pi/3 \\
\text{Region III:} \quad & 2\pi/3 \leqslant \theta \leqslant \pi.
\end{aligned}$$

a) Suppose that each particle is in the state $|1, 1\rangle$. Determine the probability with which the measurement outcome yields each region.

b) Suppose that each particle is in the state $|1, 0\rangle$. Determine the probability with which the measurement outcome yields each region.

c) Suppose that each particle is in the state $|1, -1\rangle$. Determine the probability with which the measurement outcome yields each region.

d) For which state/s will position measurement outcomes be more likely to yield an outcome near the poles alongteh z axis?

2 Wavefunctions for angular momentum eigenstates

A particle in a central potential is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} |1, 1\rangle - \frac{1}{\sqrt{2}} |1, -1\rangle$$

where $|l, m\rangle$ is an angular momentum eigenstate. The wavefunction associated with this state is

$$\psi_{lm}(r, \theta, \phi) = R(r)\, Y_{lm}(\theta, \phi)$$

where $R(r)$ depends on the precise nature of the potential and $Y_{lm}(\theta, \phi)$ is a spherical harmonic.

a) Determine an expression for the wavefunction corresponding to $|\psi\rangle$. Simplify this as much as possible (noting that it will be necessary to retain the term $R(r)$ in some form) and write this both in terms of spherical coordinates (r, θ, ϕ) and Cartesian coordinates (x, y, z).

b) Determine $\langle x \rangle, \langle y \rangle, \langle z \rangle$. (Hint: The results are very simple! You should be able to do this without evaluating any integrals explicitly. Set up the integrals and check symmetries of the integrands.)

3 Rotational Energy States of a Diatomic Molecule

In terms of rotational motion a diatomic molecule such as CO molecule can be regarded as a rigid rotator (i.e. two balls rigidly connected by a rod). Ignoring the center-of-mass and vibrational motions, the Hamiltonian for such a molecule is given by

$$\hat{H} = \frac{1}{2I}\hat{\mathbf{L}}^2$$

where I is the moment of inertia and $\hat{\mathbf{L}}^2$ represents the magnitude squared of the angular momentum.

a) Determine an expression for the energy eigenvalues for the rigid rotator. List *all* the energy eigenstates for the two lowest energy eigenvalues.

b) Now consider the same rigid rotator placed in a constant magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. The Hamiltonian is

$$\hat{H} = \frac{1}{2I}\hat{\mathbf{L}}^2 + \omega_0\hat{L}_z$$

where $\omega_0 > 0$ is a constant which depends on B_0. Verify that the *eigenstates* are the same as those for the $\omega_0 = 0$ case and determine the *eigenvalues*. List the four lowest values assuming that $\omega_0\hbar < 2\hbar^2/3I$.

c) Suppose that the emission spectrum of such a molecule were obtained. Assuming that transitions between all possible energy levels are possible, provide a list of the possible frequencies of the spectrum that arise from the four lowest energy states.