Quantum Theory I: Homework 21

Due: 28 April 2023

1 Spherical coordinate wavefunctions

Consider the following states with corresponding wavefunctions

$$|\Psi_1\rangle \leftrightarrow \Psi_1(r,\theta,\phi) = A\frac{1}{\sqrt{4\pi}}\,\frac{1}{r}e^{-r/\alpha}e^{i2\phi}$$
$$|\Psi_2\rangle \leftrightarrow \Psi_2(r,\theta,\phi) = B\frac{1}{\sqrt{4\pi}}\,e^{-r/\beta}e^{i\phi}$$

where $\alpha, \beta > 0$ are constants with units of distance.

a) Determine A and B.

b) Determine whether the states are orthogonal.

c) Suppose that a measuring device can only determine whether the particle will be found in the region where $y > 0$ or else in the region where $y < 0$. Determine whether this measuring device will yield statistically distinct outcomes for the two states. *Hint: first convert information about this region into spherical coordinates.*

d) Consider any position measuring device that gives outcomes that only refer to the direction in which a particle may be located, rather than how far it is located from the origin. Could any such measuring device yield statistically distinct outcomes for the two states?

2 Orbital angular momentum operator algebra

The orbital angular momentum operators are defined as:

$$\hat{L}_x := \hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y$$
$$\hat{L}_y := \hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z$$
$$\hat{L}_z := \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x.$$

a) Show that

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$$

Similarly one can show that (do not do these for this problem)

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$$
$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y.$$

Hint: Note that $\left[\hat{A}+\hat{B}, \hat{C}+\hat{D}\right] = \left[\hat{A}, \hat{C}\right] + \left[\hat{A}, \hat{D}\right] + \left[\hat{B}, \hat{C}\right] + \left[\hat{B}, \hat{D}\right]$.

b) Using $\hat{\boldsymbol{L}}^2 := \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ show that

$$\left[\hat{\boldsymbol{L}}^2, \hat{L}_x\right] = 0.$$

Similarly one can show that (do not do these for this problem)

$$\left[\hat{\boldsymbol{L}}^2, \hat{L}_j\right] = 0$$

for any $j = x, y, z$.

c) Show that

$$\left[\hat{L}_x, \hat{x}^2 + \hat{y}^2 + \hat{z}^2\right] = 0.$$

Similarly it can be shown that (do not do these for this problem)

$$\left[\hat{L}_j, \hat{x}^2 + \hat{y}^2 + \hat{z}^2\right] = 0$$

for $j = y, z$ and that

$$\left[\hat{L}_j, \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right] = 0$$

for any $j = x, y, z$.

d) The Hamiltonian for the isotropic three dimensional oscillator is

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right) + \frac{1}{2}m\omega^2\left(\hat{x}^2 + \hat{y}^2 + \hat{z}^2\right).$$

Show that for the isotropic harmonic oscillator

$$\left[\hat{H}, \hat{L}_j\right] = 0$$

for any $j = x, y, z$ and that

$$\left[\hat{H}, \hat{\boldsymbol{L}}^2\right] = 0.$$

What does this imply regarding the angular momentum of the isotropic harmonic oscillator?

3 Angular momentum measurements and wavefunctions

Consider a particle of mass m which is subject to measurements of its components of angular momentum. In (spherical) position space, the angular momentum operators are:

$$\hat{L}_x \leftrightarrow i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{\sin\theta} \frac{\partial}{\partial\phi} \right)$$
$$\hat{L}_y \leftrightarrow i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \frac{\cos\theta\sin\phi}{\sin\theta} \frac{\partial}{\partial\phi} \right)$$
$$\hat{L}_z \leftrightarrow -i\hbar \frac{\partial}{\partial\phi}$$

In the following we will assume that a particle is in a state which is described by a (spherical) position space wavefunction with the form

$$\Psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$$

where the radial component is normalized

$$\int_0^\infty |R(r)|^2 r^2 \mathrm{d}r = 1.$$

Suppose that the particle is in a state for which the (spherical) position space wavefunction is

$$Y(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\, e^{i\phi} \sin\theta.$$

a) Show that

$$\hat{L}_z Y(\theta,\phi) = \hbar\, Y(\theta,\phi)$$

and determine an expression for $\hat{L}_x Y(\theta,\phi)$.

b) Show that

$$\hat{L}_z^2 Y(\theta,\phi) = \hbar^2\, Y(\theta,\phi)$$

and determine an expression for $\hat{L}_x^2 Y(\theta,\phi)$.

c) Show that

$$\langle L_x \rangle = 0$$
$$\langle L_z \rangle = \hbar$$

Note that it can also be shown that $\langle L_y \rangle = 0$.

d) Show that

$$\Delta L_x = \frac{\hbar}{\sqrt{2}}$$
$$\Delta L_z = 0$$

Note that it can be shown that $\Delta L_y = \hbar/\sqrt{2}$.

e) Consider describing this state in terms of the outcome of angular momentum measurements. Is there any component of angular momentum, for which measurements yield only outcome with 100% certainty? If so what is it and what is the measurement outcome?