Quantum Theory I: Homework 19

Due: 14 April 2023

1 Transition amplitudes

A particle with mass w is in an infinite square defined by the potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

The position space wavefunction corresponding to the n^{th} energy eigenstate with energy $E_n = n^2\pi^2\hbar^2/2mL^2$ is

$$\phi_n(x) = \begin{cases} \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{n\pi x}{L}\right) & 0 \leqslant x \leqslant L \\ 0 & \text{otherwise.} \end{cases}$$

Quantities such as $\langle\phi_j|\,\hat{x}\,|\phi_k\rangle$ are useful for certain physical predictions. Show that

$$\langle\phi_j|\,\hat{x}\,|\phi_k\rangle = \begin{cases} \dfrac{4Ljk\left((-1)^{j+k}-1\right)}{\pi^2(j^2-k^2)^2} & j \neq k \\ \dfrac{L}{2} & j = k. \end{cases}$$

Hint: You can use a calculator or computer to evaluate any integrals that appear here.

2 Simple model of a radiating charged particle

Oscillating charged particles emit radiation in the form of electromagnetic waves. Electromagnetic theory predicts that the fields produced by such *dipole radiation* depends on the dipole moment of the charge and that this depends on the position of the charge.

A simple quantum model of this situation considers an ensemble of identical charged particles each with mass M in its own infinite square well with width L. For a sufficiently large ensemble, the dipole radiation emitted is identical to that of a charged particle whose position equals the expectation value for the ensemble, $\langle x\rangle(t)$. The frequencies with which this oscillates are identical to the frequencies of the emitted radiation. The aim of this exercise is to find the possible frequencies, since these can be measured readily via spectroscopic techniques. Suppose that each particle is initially in the state

$$|\Psi(0)\rangle = c_m\,|\phi_m\rangle + c_n\,|\phi_n\rangle$$

where $n > m$ and $|\phi_j\rangle$ are energy eigenstates and c_j are real constants for $j = m, n$.

a) Determine an expression for $\langle x \rangle (t)$ without evaluating terms of the form $\langle \phi_j | \hat{x} | \phi_k \rangle$. However, to simplify you expression, you can use the (easily proved) fact that $\langle \phi_j | \hat{x} | \phi_k \rangle = \langle \phi_k | \hat{x} | \phi_j \rangle$ for the infinite well.

b) The charge density is proportional to $\langle x \rangle (t)$. This contains terms that are independent of time and these amount to a fixed stationary charge distribution, which does not produce any radiation. Isolate the part of $\langle x \rangle (t)$ that oscillates with time and denote this $\langle x \rangle_{\text{osc}} (t)$. Use this to determine the *possible frequencies* with which the charges radiate.

c) The *amplitude* of the radiating fields is proportional to $\langle x \rangle_{\text{osc}} (t)$. Determine an expression for the amplitude of the dipole radiation for the situation above, without evaluating terms of the form $\langle \phi_j | \hat{x} | \phi_k \rangle$. Identify three possible situations in which no dipole radiation is emitted.

d) Although you did not need to evaluate them so far, the terms $\langle \phi_j | \hat{x} | \phi_k \rangle$ are important for dipole intensities. Evaluate these and use them to determine conditions on the initial state $|\Psi(0)\rangle$ under which no dipole radiation is emitted. Does this affect the list of possible frequencies from part b)? If so, how?

e) Provide a schematic sketch using an energy level diagram which indicates the possible frequencies for states involving the five lowest energy levels, i.e. $1 \leqslant m < n \leqslant 5$. This selection of particular frequencies is an example of *selection rules* which are widespread in atomic physics.

3 Operator relationships for the harmonic oscillator

Let $\hat{a}$ and $\hat{a}^\dagger$ denote the creation and annihilation operators for a harmonic oscillator with mass m and frequency ω.

a) Show that

$$\left[\hat{a}^\dagger \hat{a}, \hat{a}\right] = -\hat{a}$$
$$\left[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger\right] = \hat{a}^\dagger$$

b) Show that

$$\left[\hat{H}, \hat{a}\right] = -\hbar\omega\hat{a}$$
$$\left[\hat{H}, \hat{a}^\dagger\right] = \hbar\omega\hat{a}^\dagger$$

where $\hat{H}$ is the Hamiltonian for the harmonic oscillator.