Quantum Theory I: Homework 17

Due: 7 April 2023

1 Momentum wavefunction for a sharply localized moving particle

Suppose that the position wavefunction for a particle at one instant is

$$\Psi(x) = \begin{cases} \dfrac{1}{\sqrt{L}}\, e^{ip_0 x/\hbar} & \text{if } -L/2 \leqslant x \leqslant L/2 \\ 0 & \text{otherwise} \end{cases}$$

where $L > 0$ has units of length and p_0 has units of momentum.

a) Determine the position probability density.
b) Determine the position expectation value and uncertainty.
c) Determine the momentum space wavefunction.
d) Determine an expression for the momentum space probability density. How would you describe the momentum of this particle?

2 Distinguishing states by position or momentum measurements

Suppose that you are given particles which are known to be in one of the two states $|\Psi_1\rangle$ or $|\Psi_2\rangle$ where

$$\Psi_1(x) = \begin{cases} \dfrac{1}{\sqrt{L}}\, e^{ip_1 x/\hbar} & \text{if } -L/2 \leqslant x \leqslant L/2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Psi_2(x) = \begin{cases} \dfrac{1}{\sqrt{L}}\, e^{ip_2 x/\hbar} & \text{if } -L/2 \leqslant x \leqslant L/2 \\ 0 & \text{otherwise} \end{cases}$$

where $L > 0$ and $p_1 < p_2$ have units of momentum.

a) Suppose that you are given a single particle in state $|\Psi_1\rangle$ or $|\Psi_2\rangle$. Describe how likely it is that you would be able to determine the state of the particle using a position measurement. Describe how likely it is that you would be able to determine the state using a momentum measurement.
b) Suppose that you are given many copies of the particle and all are either in state $|\Psi_1\rangle$ or all in teh state $|\Psi_2\rangle$. Describe how likely it is that you would be able to determine the state using a position measurement. Describe how likely it is that you would be able to determine the state of the particle using a momentum measurement.

3 Position and momentum representations: infinite well

A particle of mass m is in an infinite square well with potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

Suppose that a particle is in energy eigenstate, $|\phi_n\rangle$.

a) Show that the momentum space wavefunction, $\tilde{\phi}_n(p)$ corresponding to $|\phi_n\rangle$ is

$$\tilde{\phi}_n(p) = \frac{\hbar^{3/2}\sqrt{\pi L}}{p^2 L^2 - n^2\pi^2\hbar^2}\, n \left[(-1)^n e^{-iLp/\hbar} - 1\right].$$

b) Find an expression for the probability distribution for momentum outcomes, $\tilde{P}(p)$. In order to facilitate graphing, it will be convenient to express this in terms of a dimensionless scaled momentum $p_{\text{s}} := pL/\hbar\pi$. Rewrite your expression for $\tilde{P}$ in terms of p_{s}.

c) Graph $\tilde{P}$ as a function of p_{s} for $n = 1, 2$ and $n = 20$. Be sure to extend the range of your graph so that it includes regions in which $\tilde{P}$ is significantly different from 0. For each graph indicate the locations of the most likely outcomes of momentum measurements and the expectation value of momentum measurements (do this first in terms of p_{s} and then in terms of p).

d) For $n = 20$ list the most likely outcomes of momentum measurements in terms of p. Calculate, using classical mechanics, the energies of a particle with each of these momenta and compare the result to the energy of the $n = 20$ eigenstate.

Hint: You should use a calculator or computer to evaluate any integrals that appear here.