Quantum Theory I: Homework 16

Due: 4 April 2023

1 Infinite well: eigenstate expansion

A particle with mass m is in an infinite well with potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

At an initial instant the wavefunction associated with the state of the particle is

$$\Psi(x) = \sqrt{\frac{30}{L^5}}\, x(x-L)$$

for $0 \leq x \leq L$. This can be represented as a superposition of energy eigenstates

$$|\Psi\rangle = \sum_{n=1}^{\infty} c_n \, |\phi_n\rangle \, .$$

a) Determine c_n for all $n = 1, 2, \ldots ...$

b) The energy is measured at the initial instant. List the possible measurement outcome for measurements of the particle's energy and the probabilities with which they occur.

c) Suppose that the energy is measured at a later instant. Will the outcomes and the probabilities with which they occur differ from the initial instant? Explain your answer.

2 Infinite well: position and energy measurements

A particle with mass m is in an infinite well with potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

Suppose that at an initial moment the state of the particle, $|\Psi\rangle$, corresponds to wavefunction

$$\Psi(x) = \begin{cases} Ax \sin\left(\dfrac{2\pi x}{L}\right) & \text{if } 0 \leqslant x \leqslant L \\ 0 & \text{otherwise.} \end{cases}$$

a) The position of the particle is measured. Will the probability with which the outcome is in the left half of the well be larger than or smaller than the probability with which the outcome is in the right half of the well? Explain your answer.

b) The energy of the particle is measured and immediately after this the position is measured. Will the probability with which the outcome is in the left half of the well be larger than or smaller than the probability with which the outcome is in the right half of the well? Explain your answer.

3 Infinite well: energy measurements

A particle with mass m is in an infinite well with potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

At a particular instant the state of the particle, $|\Psi\rangle$, corresponds to

$$\Psi(x) = \begin{cases} A\sqrt{\frac{2}{L}}x \sin\left(\frac{4\pi x}{L}\right) & \text{if } 0 \leqslant x \leqslant L \\ 0 & \text{otherwise.} \end{cases}$$

where A is a constant. The energy of the particle is measured. Determine the most likely outcome.

4 Particle in infinite well: position oscillations

A particle with mass m is in an infinite well with potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

At an initial instant, the particle could be in one of the following states:

$$\begin{aligned}
|\Psi_A(0)\rangle &= \frac{1}{\sqrt{2}}\left[|\phi_1\rangle + |\phi_2\rangle\right] \\
|\Psi_B(0)\rangle &= \frac{1}{5}\left[3\,|\phi_1\rangle + 4\,|\phi_2\rangle\right] \\
|\Psi_C(0)\rangle &= \frac{1}{\sqrt{2}}\left[|\phi_1\rangle + |\phi_3\rangle\right] \\
|\Psi_D(0)\rangle &= \frac{1}{\sqrt{2}}\left[|\phi_2\rangle + |\phi_3\rangle\right] \\
|\Psi_E(0)\rangle &= \frac{1}{\sqrt{2}}\left[|\phi_2\rangle - |\phi_3\rangle\right]
\end{aligned}$$

These subsequently evolve as time passes. In each case the position probability density and the position expectation value oscillate. Rank these in order of increasing frequency of oscillation.

5 Time evolution of a wavefunction in a square well

A particle with mass m is in an infinite well with potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

At an initial instant ($t = 0$), the particle is known to be in the quarter of the well $L/4 \leqslant x \leqslant L/2$. All locations within this range are equally likely.

The aim of this exercise is to determine how this evolves with time.

a) The initial state of of the particle can be expressed as

$$\Psi(x, 0) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

where $\phi_n(x)$ is the energy eigenstate with energy $E_n = n^2\pi^2\hbar^2/2mL^2$. Determine c_n.

b) Plot

$$\sum_{n=1}^{10} c_n \phi_n(x)$$

and verify that it approximates $\Psi(x, 0)$. Set $L = 1$ to do the plot.

c) Determine an expression for $\Psi(x, t)$ as a superposition of energy eigenstates.

d) In order to do any plotting the constants must be removed. Define a rescaled dimensionless time via

$$t' := t\frac{\pi^2\hbar.}{2mL^2}$$

and determine an expression for $\Psi(x, t')$ where $L = 1$.

e) Truncate the series $n = 10$ and plot the position probability density at $t' = 0$, $t' = 1.05$, $t' = 2.1$, and $t' = 3.2$. How would you describe the location of the particle qualitatively at each of these moments?

f) Create an animation to view the position probability density as it evolves for $0 \leqslant t' \leqslant 3$.