Quantum Theory I: Homework 15

Due: 31 March 2023

1 Gaussian wavefunction

Consider a particle in one dimension in the state $|\Psi\rangle$ with associated wavefunction

$$\Psi(x) = Ae^{-(x-x_0)^2/2\alpha^2}$$

where x_0 is real and $\alpha > 0$. In the following you can use software such as Mathematica or Wolfram Alpha to evaluate integrals symbolically.

a) Determine the normalization constant A.

b) Graph the position probability density. How do x_0 and α affect the graph? How would they affect the outcomes of position measurements?

c) Suppose that many copies of the particle are prepared in the same state $|\Psi\rangle$ and the position of each is measured. The results are averaged. Use the state to predict what the average should be. *Note that this is inherently a statistical prediction.*

d) If this entire process were repeated the average would fluctuate. The uncertainty quantifies the extent of the fluctuations. Determine the uncertainty in the position measurement outcomes.

2 Infinite well: wavefunctions and position measurements

To illustrate the meaning of a typical wavefunction, consider position measurements on a particle in an infinite square well for which the potential is:

$$V(x) = \begin{cases} 0 & 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

Assume that the particle is in an energy eigenstate, $\phi_n(x,t)$ where $n = 1, 2, 3, \ldots$. This exercise should show that the wavefunction is closely related statistically to outcomes of position measurements. However, it should emerge that the relationship between energy eigenstates and position measurements is seldom very sharp.

a) Show that the probability density for position measurement outcomes, $P_n(x,t)$, is independent of time and use this to determine whether the time at which you measure the particle's position will influence the outcome of a position measurement.

b) Suppose that you measure the positions of 12000 independent particles, each prepared in the energy eigenstate $\phi_n(x,t)$. On average, for how many of the particles do you expect an outcome in the region $0 \leqslant x \leqslant L/6$? Does this depend on the eigenstate?

c) Any position measuring device will have a finite resolution. Suppose that the apparatus for measuring the position of the particle is so crudely constructed that it can only determine whether the position measurement outcome is in one of the following regions:

$$\begin{array}{ll} \text{Region I:} & 0 \leqslant x \leqslant L/4 \\ \text{Region II:} & L/4 \leqslant x \leqslant L/2 \\ \text{Region III:} & L/2 \leqslant x \leqslant 3L/4 \\ \text{Region IV:} & 3L/4 \leqslant x \leqslant L \end{array}$$

Given that the particle is in the energy eigenstate $_n(x, t)$, determine the probability that the position measurement outcome gives an outcome in each of the regions (i.e. calculate the probability that the position measurement outcome is somewhere in region I, and repeat this for regions II,III and IV.). *Hint: It may appear that you will have to calculate four integrals, one for each region. This is not necessary; try to solve the problem with a minimal number of integrations by exploiting various symmetries in the probability density.*

d) Suppose your lab assistant has two boxes, each containing one particle in an infinite square well potential, and in one box he prepares the particle in the $n = 4$ energy eigenstate and in the other the $n = 5$ energy eigenstate. He keeps a secret record of the states in each box. He hands you one and you are required to determine the state of the particle in the box using your crude position measuring apparatus of part (c). Can you determine the state of the particle **with certainty** just using your apparatus?

e) Suppose that your lab assistant has two boxes each containing 12000 particles and in one prepares **all the particles** in the $n = 4$ energy eigenstate and in the other **all the particles** in the $n = 5$ energy eigenstate. He again hands you one box and asks you to determine the state of the particles in the box by only using the crude apparatus. Statistically, can you determine which box he gave you? Explain your answer.

f) Continuing the game between you and your lab assistant, suppose that he chose to use particles in the $n = 2$ or $n = 4$ energy eigenstates. If the boxes each contained 12000 particles, would you be able to distinguish them? Would any refinement of your measuring device help? Motivate your answer.

3 Infinite well: superposition of energy eigenstates

Suppose that, at one instant, a particle of mass m in an infinite square well of width L is in the state

$$\Psi(x) = \frac{1}{2}\,\phi_1(x) + \frac{1}{3}\,e^{i\pi/2}\,\phi_3(x) + \frac{1}{2}\,e^{i\pi}\,\phi_4(x) + \sqrt{\frac{7}{18}}\,e^{i\pi/2}\,\phi_6(x)$$

where $\phi_n(x)$ are the energy eigenstates.

a) Suppose that the energy of a single particle in this state is measured. What are the possible outcomes of this measurement?

b) Is it possible that the energy measurement yields $E = 5\,\dfrac{\hbar^2\pi^2}{2mL^2}$?

c) What are the probabilities with which the various outcomes of the energy measurement occur?

d) Suppose that 10000 identical particles are each prepared in the this state and that the energy of each is measured. List the outcomes of this measurement and the number of times that you expect to get each outcome. Determine the expectation value of the energy and the uncertainty in the energy.

4 Particle in an infinite well: superposition states

Suppose that, at one instant, a particle of mass m in an infinite square well of width L is in one of the states

$$\begin{aligned}
\Psi_A(x) &= \frac{1}{\sqrt{2}}\,\phi_1(x) + \frac{1}{\sqrt{2}}\,\phi_2(x)\\
\Psi_B(x) &= \frac{1}{\sqrt{2}}\,\phi_1(x) - \frac{1}{\sqrt{2}}\,\phi_2(x)\\
\Psi_C(x) &= \frac{1}{\sqrt{2}}\,\phi_1(x) + \frac{i}{\sqrt{2}}\,\phi_2(x)\\
\Psi_D(x) &= \frac{1}{\sqrt{2}}\,\phi_1(x) - \frac{i}{\sqrt{2}}\,\phi_2(x)
\end{aligned}$$

where $\phi_n(x)$ are the energy eigenstates.

a) The energy of the particle is measured. Could you use the outcome of the measurement to decide which state the particle is in with certainty? Could you use the outcome to decide which state is more likely than the others? Explain your answer.

b) The position of the particle is measured. Could you use the outcome of the measurement to decide which state the particle is in with certainty? Could you use the outcome to decide which state is more likely than the others? Explain your answer.

c) Which collections of these states are perfectly distinguishable by *some* measurement? Explain your answer.