Phys 321
Spring 2023

Quantum Theory I: Homework 14

Due: 28 March 2023

1 Dirac delta function

a) Apply the definition of the Dirac delta function to evaluate

$$\int_{-\infty}^{\infty} x^2 \delta(x - x_0)\, \mathrm{d}x$$

where x_0 is any real number.

b) An approximate representation of the Dirac delta function is

$$\delta_\epsilon(x - x_0) := \begin{cases} 0 & \text{if } x < x_0 - \epsilon/2 \\ \dfrac{1}{\epsilon} & \text{if } x_0 - \epsilon/2 < x < x_0 + \epsilon/2 \\ 0 & \text{if } x > x_0 + \epsilon/2. \end{cases}$$

where the notion is that

$$\lim_{\epsilon \to 0} \delta_\epsilon(x - x_0) = \delta(x - x_0).$$

Integrate

$$\int_{-\infty}^{\infty} x^2 \delta_\epsilon(x - x_0)\, \mathrm{d}x$$

and verify that as $\epsilon \to 0$ the result is the same as that of the previous part.

c) Evaluate

$$\int_{-\infty}^{\infty} 3x^2 \, \sin x \, \delta(2x - \pi)\, \mathrm{d}x.$$

Hint: Perform a u substitution so that the delta function appears as $\delta(u)$ in the integrand and then use the definition of the delta function to evaluate the integral.

2 Wavefunctions on a restricted range

Suppose that

$$|\Psi_1\rangle \leftrightarrow \Psi_1(x) := \begin{cases} B(x^2 - a^2) & \text{if } -a \leqslant x \leqslant a \\ 0 & \text{if } |x| \geqslant a \end{cases}$$

where $a > 0$. Similarly let

$$|\Psi_2\rangle \leftrightarrow \Psi_2(x) := \begin{cases} Cx(x^2 - a^2) & \text{if } -a \leqslant x \leqslant a \\ 0 & \text{if } |x| \geqslant a \end{cases}$$

a) Determine expressions for the normalization constants B and C.

b) Show that $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are orthogonal.

In your solutions you should show how to set up the relevant integrals by hand. The integrals themselves can be evaluated using a calculator or software.

3 Superpositions of wavefunctions

Let $\Psi_1(x)$ and $\Psi_2(x)$ represent any wavefunctions that are orthonormal. Then consider

$$\Phi_1(x) := \frac{1}{\sqrt{2}}\Psi_1(x) + \frac{i}{\sqrt{2}}\Psi_2(x)$$
$$\Phi_2(x) := \frac{1}{\sqrt{2}}\Psi_1(x) - \frac{i}{\sqrt{2}}\Psi_2(x)$$

Show that these are orthonormal.

4 Inner product between states for particles in one dimension

Consider the following states and corresponding wavefunctions for particles in one dimension.

$$|\Psi_1\rangle \leftrightarrow \Psi_1(x) = \left(\frac{1}{a\pi}\right)^{1/4} e^{-x^2/2a} \quad \text{and}$$
$$|\Psi_2\rangle \leftrightarrow \Psi_2(x) = \left(\frac{4}{a^3\pi}\right)^{1/4} x\, e^{-x^2/2a}$$

where $a > 0$.

a) Show that $\langle\Psi_i|\Psi_j\rangle = \delta_{ij}$ for all combinations of i and j.

b) Let

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,|\Psi_1\rangle + \frac{1}{\sqrt{2}}\,|\Psi_2\rangle \quad \text{and}$$
$$|\Phi\rangle = \frac{3}{5}\,|\Psi_1\rangle + \frac{4i}{5}\,|\Psi_2\rangle$$

where $a > 0$. Show that $\langle\Psi|\Psi\rangle = 1$ and $\langle\Phi|\Phi\rangle = 1$ and determine $\langle\Phi|\Psi\rangle$. *Hint: Try to do these without computing any integrals.*

5 Gaussian wavefunctions and localized particles

Consider the following states and corresponding wavefunctions for particles (labeled A and B) in one dimension.

$$|\Psi_{\rm A}\rangle \leftrightarrow \Psi_{\rm A}(x) = \left(\frac{1}{\pi}\right)^{1/4} e^{-x^2/2} \quad \text{and}$$
$$|\Psi_{\rm B}\rangle \leftrightarrow \Psi_{\rm B}(x) = \left(\frac{1}{\pi}\right)^{1/4} e^{-(x-x_0)^2/2}$$

where x_0 is real. These are normalized.

a) Plot the probability density $P_{\mathrm{A}}(x)$ over the range $-8 \leqslant x \leqslant 8$ for particle A. How would you describe its location qualitatively?

b) Plot the probability density $P_{\mathrm{B}}(x)$ over the range $-8 \leqslant x \leqslant 8$ for particle B for the case where $x_0 = 5$. How would you describe its location qualitatively? Can its location be said to be different to that for particle A? Explain your answer.

c) Plot the probability density $P_{\mathrm{B}}(x)$ over the range $-8 \leqslant x \leqslant 8$ for particle B for the case where $x_0 = 2$. How would you describe its location qualitatively? Can its location be said to be different to that for particle A? Explain your answer.

d) Is it possible that two different particles have states that allows one to identify the particles by their location? Explain your answer.