Quantum Theory I: Homework 10

Due: 28 February 2023

1 Transformations of states

Suppose that a spin-1/2 particle passes through a region containing various magnetic fields and it is determined that the corresponding evolution operator is

$$\hat{U} = |+\hat{\boldsymbol{y}}\rangle \langle +\hat{\boldsymbol{x}}| + |-\hat{\boldsymbol{y}}\rangle \langle -\hat{\boldsymbol{x}}|$$

where:

$$|+\hat{\boldsymbol{y}}\rangle = \frac{1}{\sqrt{2}} \Big(|+\hat{\boldsymbol{z}}\rangle + i \, |-\hat{\boldsymbol{z}}\rangle \Big)$$
$$|-\hat{\boldsymbol{y}}\rangle = \frac{1}{\sqrt{2}} \Big(|+\hat{\boldsymbol{z}}\rangle - i \, |-\hat{\boldsymbol{z}}\rangle \Big).$$

a) Determine the matrix representation of $\hat{U}$ in the $\{|+\hat{\boldsymbol{z}}\rangle, |-\hat{\boldsymbol{z}}\rangle\}$ basis.

b) Verify that $\hat{U}$ is unitary.

c) Suppose that a particle is in the state $|+\hat{\boldsymbol{x}}\rangle$ immediately before passing through this region. Determine the state of the particle immediately after passing through this region.

d) Suppose that a particle is in the state $|-\hat{\boldsymbol{x}}\rangle$ immediately before passing through this region. Determine the state of the particle immediately after passing through this region.

e) Suppose that a particle is in the state $|+\hat{\boldsymbol{z}}\rangle$ immediately before passing through this region. Determine the state of the particle immediately after passing through this region. Determine the probability with which a subsequent SG $\hat{\boldsymbol{z}}$ measurement would yield $S_z = +\hbar/2$.

f) Suppose that a particle is in the state $|-\hat{\boldsymbol{z}}\rangle$ immediately before passing through this region. Determine the state of the particle immediately after passing through this region. Determine the probability with which a subsequent SG $\hat{\boldsymbol{z}}$ measurement would yield $S_z = +\hbar/2$.

g) This operation can be viewed as a rotation. Based on your answers above, about which axis and through which angle does this rotation take place.

2 Unitary operators

Consider the operators

$$\hat{U}_1 := \frac{1}{5} \begin{pmatrix} 3 & 4i \\ 4i & 3 \end{pmatrix} \qquad \text{and} \qquad \hat{U}_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Which of these represent possible evolutions for quantum states? Explain your answer.

3 Combinations of evolution operators

Consider the evolution operators

$$\hat{U}_1 := \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \qquad \text{and} \qquad \hat{U}_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

a) Suppose that a particle emerges from an SG $\hat{\boldsymbol{x}}$ measurement with outcome $+\hbar/2$. It is then subjected to evolution represented by $\hat{U}_1$. Determine the state of the particle after this.

b) Subsequently the particle is subjected to evolution represented by $\hat{U}_2$. Determine the state of the particle after this.

c) After the pair of evolutions, a component of spin is measured. In what direction should the measurement be so that one outcome occurs with certainty?

4 Evolution plus measurement

Consider a spin-1/2 particle in an initial state $|+\hat{\boldsymbol{z}}\rangle$. Suppose that the system is subjected to external influences described by a unitary evolution operator $\hat{U}$. After these influences a measurement of S_z is performed.

a) Will this measurement always yield one particular outcome with certainty?

b) Are there some situations where this measurement will yield one particular outcome with certainty? If so, describe these.

c) Are there some situations where this measurement will not yield one particular outcome with certainty? If so, describe these.