Quantum Theory I: Homework 9

Due: 24 February 2023

1 Spin commutation relations

Show that

$$\begin{aligned}\left[\hat{S}_x, \hat{S}_y\right] &= i\hbar\hat{S}_z \\ \left[\hat{S}_y, \hat{S}_z\right] &= i\hbar\hat{S}_x \\ \left[\hat{S}_z, \hat{S}_x\right] &= i\hbar\hat{S}_y\end{aligned}$$

What do these say about the existence of a state such that measurements of more than one component of spin will yield an outcome with certainty for each of the components of spin?

2 Commutation Relations

Let

$$\begin{aligned}\hat{A} &= \cos\varphi\,\hat{S}_x + \sin\varphi\,\hat{S}_y \\ \hat{B} &= \frac{1}{\sqrt{2}}\,\hat{S}_x + \frac{1}{\sqrt{2}}\,\hat{S}_y\end{aligned}$$

where φ is real.

a) Show that $\hat{A}$ and $\hat{B}$ are each Hermitian.

b) Determine an expression for $\left[\hat{A}, \hat{B}\right]$.

c) For what values of φ are these two observables compatible?

3 States and measurements

Consider the states

$$\begin{aligned}|\phi_1\rangle &= \frac{3}{5}\,|+\hat{\boldsymbol{z}}\rangle + \frac{4}{5}\,|-\hat{\boldsymbol{z}}\rangle \\ |\phi_2\rangle &= \frac{3}{5}\,|+\hat{\boldsymbol{z}}\rangle + \frac{4i}{5}\,|-\hat{\boldsymbol{z}}\rangle \\ |\phi_3\rangle &= \frac{3}{5}\,|+\hat{\boldsymbol{z}}\rangle - \frac{4i}{5}\,|-\hat{\boldsymbol{z}}\rangle \\ |\phi_4\rangle &= \frac{4}{5}\,|+\hat{\boldsymbol{z}}\rangle - \frac{3}{5}\,|-\hat{\boldsymbol{z}}\rangle \\ |\phi_5\rangle &= \frac{4}{5}\,|+\hat{\boldsymbol{z}}\rangle + \frac{3i}{5}\,|-\hat{\boldsymbol{z}}\rangle \\ |\phi_6\rangle &= \frac{4}{5}\,|+\hat{\boldsymbol{z}}\rangle - \frac{3i}{5}\,|-\hat{\boldsymbol{z}}\rangle\end{aligned}$$

a) For each state, is there some measurement such that it will yield one outcome (i.e. $\pm\hbar/2$) with certainty when performed on a particle in that state? Explain your answer.

b) For some groups of these states there may be a single measurement such that the measurement will yield distinct outcomes when performed on each state in the group. Group the states according to this.

c) Construct the observables for each of these measurements.

d) Explain how you could use these observables to determine whether the associated measurements are compatible.