Phys 321
Spring 2023

Quantum Theory I: Class Exam 1

2 March 2023

Name: SOLUTION **Total:** **/50**

Instructions

- There are 6 questions on 10 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Planck's constant	$h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $\quad \hbar = 1.05 \times 10^{-34}\,\mathrm{Js}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\,\mathrm{kg} = 511 \times 10^{3}\,\mathrm{eV/c^2}$
Mass of proton	$m_p = 1.673 \times 10^{-27}\,\mathrm{kg} = 938.3 \times 10^{6}\,\mathrm{eV/c^2}$
Mass of neutron	$m_n = 1.675 \times 10^{-27}\,\mathrm{kg} = 939.6 \times 10^{6}\,\mathrm{eV/c^2}$
Spherical coordinates	$\hat{\mathbf{n}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$
Spin 1/2 state	$\lvert +\hat{n}\rangle = \cos(\theta/2)\lvert +\hat{z}\rangle + e^{i\phi}\sin(\theta/2)\lvert -\hat{z}\rangle$
Spin 1/2 state	$\lvert -\hat{n}\rangle = \sin(\theta/2)\lvert +\hat{z}\rangle - e^{i\phi}\cos(\theta/2)\lvert -\hat{z}\rangle$
Euler relation	$e^{i\alpha} = \cos\alpha + i\sin\alpha$
Spin observables	$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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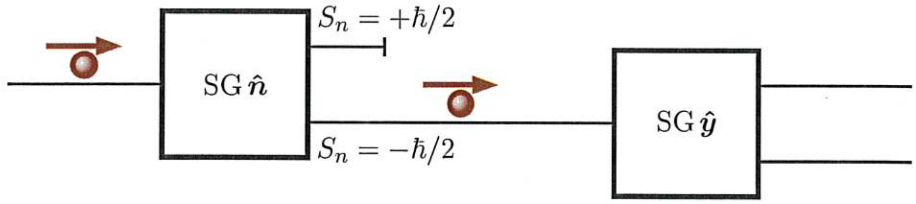
Question 1

A spin-1/2 particle is subjected to an SG $\hat{\boldsymbol{n}}$ apparatus, where $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}\hat{\mathbf{y}} + \frac{1}{\sqrt{2}}\hat{\mathbf{z}}$, and emerges in the output for which $S_n = -\hbar/2$.



a) The particle is subsequently subjected to an SG $\hat{\boldsymbol{y}}$ apparatus. List the measurement outcomes and the probabilities with which they occur, given that the particle does emerge with $S_n = -\hbar/2$ from the first measurement.

After SG$\hat{n}$ the state is $|-\hat{n}\rangle$. Here for $\hat{n}$

$\theta = \pi/4 \quad \phi = \pi/2$

$$|-\hat{n}\rangle = \sin\left(\tfrac{\theta}{2}\right)|+\hat{z}\rangle - e^{i\phi}\cos\left(\tfrac{\theta}{2}\right)|-\hat{z}\rangle$$

$$= \sin\left(\tfrac{\pi}{8}\right)|+\hat{z}\rangle - \underbrace{e^{i\pi/2}}_{i}\cos\left(\pi/8\right)|-\hat{z}\rangle \Rightarrow |-\hat{n}\rangle = \sin\tfrac{\pi}{8}|+\hat{z}\rangle - i\cos\tfrac{\pi}{8}|-\hat{z}\rangle$$

Then

outcomes	probability
$S_y = +\hbar/2$	$\lvert\langle +\hat{y} \vert -\hat{n}\rangle\rvert^2$
$S_y = -\hbar/2$	$\lvert\langle -\hat{y} \vert -\hat{n}\rangle\rvert^2$

where

$$|+\hat{y}\rangle = \tfrac{1}{\sqrt{2}}|+\hat{z}\rangle + \tfrac{i}{\sqrt{2}}|-\hat{z}\rangle$$

$$|-\hat{y}\rangle = \tfrac{1}{\sqrt{2}}|+\hat{z}\rangle - \tfrac{i}{\sqrt{2}}|-\hat{z}\rangle$$

Thus $$\langle +\hat{y}|-\hat{n}\rangle = \tfrac{1}{\sqrt{2}}(1 \;\; -i)\begin{pmatrix}\sin\pi/8 \\ -i\cos\pi/8\end{pmatrix} = \tfrac{1}{\sqrt{2}}\left(\sin\pi/8 - \cos\pi/8\right)$$

Question 1 continued ...

Then $\langle -\hat{y} | -\hat{n} \rangle = \frac{1}{\sqrt{2}}(1 \ +i)\begin{pmatrix} \sin\pi/8 \\ -i\cos\pi/8 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\sin\pi/8 + \cos\pi/8\right).$

So

$$\text{Prob}\left(S_y = +\hbar/2\right) = \left|\frac{1}{\sqrt{2}}\left(\cos\pi/8 - \sin\pi/8\right)\right|^2 = \frac{1}{2}\left[\cos^2\pi/8 + \sin^2\frac{\pi}{8} - 2\cos\frac{\pi}{8}\sin\pi/8\right]$$
$$= \frac{1}{2}\left[1 - \cos\frac{\pi}{4}\right] = \frac{1}{2}\left[1 - \frac{1}{\sqrt{2}}\right]$$

$$\text{Prob}\left(S_y = -\hbar/2\right) = \left|\frac{1}{\sqrt{2}}\left(\cos\pi/8 + \sin\pi/8\right)\right|^2 = \cdots \ \frac{1}{2}\left[1 + \cos\frac{\pi}{4}\right] = \frac{1}{2}\left[1 + \frac{1}{\sqrt{2}}\right]$$

b) Suppose that, prior to the SG $\hat{n}$ apparatus on the left, the state of the particle was $|+\hat{y}\rangle$. Will the measurement outcome from the SG $\hat{y}$ apparatus on the right be $S_y = +\hbar/2$ with certainty? Would your answer be any different if the SG $\hat{n}$ apparatus on the left were not present? Explain your answers.

→ **No**, because the state emerging from the right is $|-\hat{n}\rangle$ and part a) shows we do not get $S_y = +\hbar/2$ with certainty

If the SG$\hat{n}$ apparatus were absent then the state prior to SG$\hat{y}$ would be $|+\hat{y}\rangle$. This does give $S_y = +\hbar/2$ with certainty. It *is* different.

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Question 2

A spin-1/2 particle is in the state

$$|\Psi\rangle = \frac{4}{5}\,|+\hat{z}\rangle - \frac{3i}{5}\,|-\hat{z}\rangle\,.$$

a) Determine the direction, $\hat{\mathbf{n}}$, such that a measurement of S_n will yield $S_n = +\hbar/2$ with certainty.

$$|+\hat{n}\rangle = \cos\frac{\theta}{2}|+\hat{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\hat{z}\rangle$$

$$\cos\frac{\theta}{2} = \frac{4}{5} \Rightarrow \frac{\theta}{2} = \arccos\frac{4}{5} = 0{,}64\,\text{rad} \Rightarrow \theta = 1{,}29\,\text{rad}$$

$$\sin\frac{\theta}{2}\,e^{i\phi} = \frac{-3i}{5} \Rightarrow \sin\left(\frac{1{,}29}{2}\right)e^{i\phi} = -\frac{3i}{5}$$

$$\Rightarrow 0{,}6\,e^{i\phi} = -\frac{3}{5}i \Rightarrow e^{i\phi} = -i$$

so $\theta = 1{,}29\,\text{rad}$ $\phi = 3\pi/2$

$$\Rightarrow \phi = 3\pi/2$$

b) Determine the state of the system, expressed in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$, such that a measurement of S_n will yield $S_n = -\hbar/2$ with certainty.

$|-\hat{n}\rangle$ will yield $S_n = -\hbar/2$ with certainty

$$|-\hat{n}\rangle = \sin\frac{\theta}{2}|+\hat{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\hat{z}\rangle$$

$$= \frac{3}{5}|+\hat{z}\rangle - \underbrace{e^{i3\pi/2}}_{-i}\frac{4}{5}|-\hat{z}\rangle$$

$$\Rightarrow |-\hat{n}\rangle = \frac{3}{5}|+\hat{z}\rangle + i\frac{4}{5}|-\hat{z}\rangle$$

/8

Question 3

Consider a collection of spin-1/2 particles. Each particle is subjected to a SG $\hat{\boldsymbol{x}}$ measurement.

a) Using

$$\hat{S}_n = \frac{\hbar}{2}\left|+\hat{\boldsymbol{n}}\right\rangle\left\langle+\hat{\boldsymbol{n}}\right| - \frac{\hbar}{2}\left|-\hat{\boldsymbol{n}}\right\rangle\left\langle-\hat{\boldsymbol{n}}\right|,$$

verify that the matrix representation of $\hat{S}_x$ in the $\{\left|+\hat{\boldsymbol{z}}\right\rangle, \left|-\hat{\boldsymbol{z}}\right\rangle\}$ basis is

$$\hat{S}_x \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\hat{S}_x = \frac{\hbar}{2}\left|+\hat{x}\right\rangle\left\langle+\hat{x}\right| - \frac{\hbar}{2}\left|-\hat{x}\right\rangle\left\langle-\hat{x}\right|$$

Then $\left|+\hat{x}\right\rangle = \frac{1}{\sqrt{2}}\left|+\hat{z}\right\rangle + \frac{1}{\sqrt{2}}\left|-\hat{z}\right\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\left|-\hat{x}\right\rangle = \frac{1}{\sqrt{2}}\left|+\hat{z}\right\rangle - \frac{1}{\sqrt{2}}\left|-\hat{z}\right\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\hat{S}_x \rightsquigarrow \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\hbar}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Question 3 continued ...

$|+\hat{x}\rangle$

b) Suppose that 2/5 of the particles are in a subcollection with the state $|+\cancel{\hat{x}}\,\hat{y}\rangle$ and 3/5 of the particles are in a subcollection with the state $\cancel{|+\hat{x}\rangle}\ \hat{y}$. Determine the expectation value of S_x for each subcollection and *also* the expectation value of S_x for the entire collection.

Subcollection $|+\hat{y}\rangle$ gives: $\qquad |+\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$

$$\langle S_x\rangle = \langle +\hat{y}|S_x|+\hat{y}\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ +i\end{pmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{\hbar}{4}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}i\\ 1\end{pmatrix} = 0$$

Subcollection $|+\hat{x}\rangle$ gives

$$\langle S_x\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{\hbar}{2}$$

For the entire collection

$$\langle S_x\rangle = \frac{2}{5}\,0 + \frac{3}{5}\frac{\hbar}{2} \Rightarrow \langle S_x\rangle = \frac{3\hbar}{10}$$

/12

Question 4

Consider the hypothetical observable for a spin-1/2 system

$$\hat{A} = 4\,|+\hat{x}\rangle\,\langle+\hat{x}| + 6\,|-\hat{x}\rangle\,\langle-\hat{x}|$$

It is fairly straightforward to show that

$$\begin{aligned}\hat{A}\,|+\hat{x}\rangle &= 4\,|+\hat{x}\rangle \quad \text{and}\\ \hat{A}\,|-\hat{x}\rangle &= 6\,|-\hat{x}\rangle\,.\end{aligned}$$

Is it possible that a measurement of the associated quantity could yield $A = 5$? Explain your answer.

Those are the eigenvalue equations $\hat{A}|\phi\rangle = \lambda|\phi\rangle$.

The two eigenvalues are $\lambda = 4$ and $\lambda = 6$.

The possible measurement outcomes are $A = 4$ and $A = 6$

Then $A = 5$ is impossible

/4

Question 5

Various components of spin can be measured for a spin-1/2 particle. Consider S_x and S_y. Someone claims that there might be a state of the particle such that it will yield one measurement outcome for S_x with certainty and also one, possibly different, measurement outcome for S_y with certainty. Explain *how* you could use quantum theory (not an experiment) to decide whether this statement is true or not?

This is only true if $[\hat{S}_x, \hat{S}_y] = 0$

We would have to evaluate the commutator, and see if the result is zero.

/4

Question 6

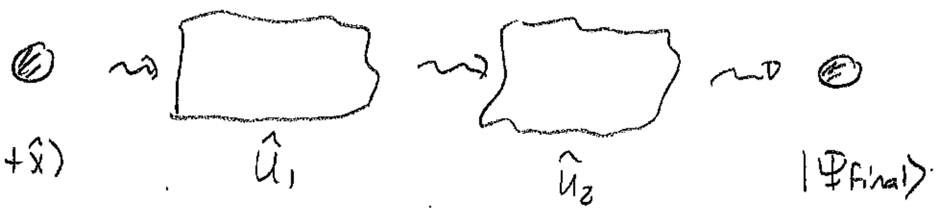
A spin-1/2 particle emerges from an SG$\hat{\boldsymbol{x}}$ apparatus in the $S_x = +\hbar/2$ output and subsequently passes through two magnetic field regions in succession. The evolution operator for the first region is

$$\hat{U}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

while that for the second region is

$$\hat{U}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

a) Determine the state of the particle after it has passed through both regions.



After $\hat{U}_1$ the state is

$$|\Psi_1\rangle = \hat{U}_1 |+\hat{x}\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\hat{x}\rangle$$

After $\hat{U}_2$ the state is:

$$\hat{U}_2 |\Psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus

$$|\Psi_f\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-\hat{z}\rangle.$$

Question 6 continued ...

b) The particle is subjected to an SG $\hat{z}$ apparatus after passing through the second region. Determine the probability with which the measurement outcome is $S_z = +\hbar/2$.

The measurement of S_z on $|\Psi_f\rangle = |-\hat{z}\rangle$ gives $S_z = -\hbar/2$ with certainty. Thus

$$\text{Prob}(S_z = -\hbar/2) = 1$$

and $\text{Prob}(S_z = +\hbar/2) = 0$

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