Phys 321
Spring 2022

Quantum Theory I: Class Exam II

21 April 2022

Name: Solution **Total:** **/50**

Instructions

- There are 5 questions on 10 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Planck's constant	$h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $\quad \hbar = 1.05 \times 10^{-34}\,\mathrm{Js}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\,\mathrm{kg} = 511 \times 10^3\,\mathrm{eV/c^2}$
Mass of proton	$m_p = 1.673 \times 10^{-27}\,\mathrm{kg} = 938.3 \times 10^6\,\mathrm{eV/c^2}$
Mass of neutron	$m_n = 1.675 \times 10^{-27}\,\mathrm{kg} = 939.6 \times 10^6\,\mathrm{eV/c^2}$
Spherical coordinates	$\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$
Spin 1/2 state	$\lvert +\hat{n}\rangle = \cos(\theta/2)\lvert +\hat{z}\rangle + e^{i\phi}\sin(\theta/2)\lvert -\hat{z}\rangle$
Spin 1/2 state	$\lvert -\hat{n}\rangle = \sin(\theta/2)\lvert +\hat{z}\rangle - e^{i\phi}\cos(\theta/2)\lvert -\hat{z}\rangle$
Euler relation	$e^{i\alpha} = \cos\alpha + i\sin\alpha$
Spin observables	$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Rep. in $\lvert\pm\hat{\mathbf{z}}\rangle$ basis	$\hat{R}(\varphi \boldsymbol{n}) = \begin{pmatrix} \cos\left(\frac{\varphi}{2}\right) - i\sin\left(\frac{\varphi}{2}\right)\cos\theta & -i\sin\left(\frac{\varphi}{2}\right)e^{-i\phi}\sin\theta \\ -i\sin\left(\frac{\varphi}{2}\right)e^{i\phi}\sin\theta & cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\cos\theta \end{pmatrix}$

Physical constants and useful formulae

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \quad \text{if } |a| < 1.$$

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$$

$$\sum_{n=0}^{\infty} n\frac{a^n}{n!} = ae^a$$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x\sin(ax)\,\mathrm{d}x = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$$

$$\int x^2\sin(ax)\,\mathrm{d}x = \frac{2x^2}{a}\sin(ax) + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos(ax)$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} xe^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{\beta\sqrt{\pi}}{2\alpha^{3/2}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} x^2e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{(\beta^2+2\alpha)\sqrt{\pi}}{4\alpha^{5/2}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} x^3e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{\beta(\beta^2+6\alpha)\sqrt{\pi}}{8\alpha^{7/2}}\,e^{\beta^2/4\alpha}$$

Question 1

A spin-1/2 particle is initially in the state $|+\hat{y}\rangle$ and it is then placed in a constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{x}}$.

a) Show that the Hamiltonian has the form

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$$\hat{H} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and determine an expression for ω in terms of the particle mass m, charge q and g-factor g.

$$\hat{H} = -\frac{gq}{2m}\hat{\vec{S}}\cdot\vec{B} = -\frac{gq}{2m}\left[\hat{S}_x \underset{B_0}{B_x} + \hat{S}_y \cancel{B_y} + \hat{S}_z \cancel{B_z}\right]$$

$$= -\frac{gqB_0}{2m}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar\omega}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \omega = \frac{-gqB_0}{2m}$$

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b) Determine an expression for the state of the particle after it has been in the field for time t.

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$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$$

Note $|\Psi(0)\rangle = |+\hat{y}\rangle$

$$= \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

$$\rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$e^{-i\hat{H}t/\hbar} = e^{-i\omega t/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\omega t}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n$$

Now $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}$

Question 1 continued ...

Thus

$$e^{-i\hat{H}t/\hbar} = \hat{I}\left[1 + \left(\frac{-i\omega t}{2}\right)^2 \frac{1}{2!} + \cdots\right]$$

$$+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left[\frac{-i\omega t}{2} + \frac{1}{3!}\left(\frac{-i\omega t}{2}\right)^3 + \cdots\right]$$

$$= \hat{I}\left(1 - \left(\frac{\omega t}{2}\right)^2 \frac{1}{2!} + \cdots\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(-i)\left[\left(\frac{\omega t}{2}\right) - \frac{1}{3!}\left(\frac{\omega t}{2}\right)^3 + \cdots\right]$$

$$= \hat{I}\cos\left(\frac{\omega t}{2}\right) - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\left(\frac{\omega t}{2}\right) = \begin{pmatrix} \cos\frac{\omega t}{2} & -i\sin\frac{\omega t}{2} \\ -i\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{pmatrix}$$

$$|\Psi(t)\rangle = \begin{pmatrix} \cos\frac{\omega t}{2} & -i\sin\frac{\omega t}{2} \\ -i\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\frac{\omega t}{2} + \sin\frac{\omega t}{2} \\ i\left(\cos\frac{\omega t}{2} - \sin\frac{\omega t}{2}\right) \end{pmatrix}$$

c) Determine an expression, in terms of ω for the time should pass after the particle entered the field until it first reaches a state where an SG $\hat{z}$ measurement yields $S_z = \hbar/2$ with certainty.

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We need $\text{Prob}(S_z = \hbar/2) = 1$

$$\Rightarrow \quad |\langle +z|\Psi(t)\rangle|^2 = 1$$

$$\langle +\hat{z}|\Psi(t)\rangle = (1 \;\; 0)\begin{pmatrix} \cos\frac{\omega t}{2} + \sin\frac{\omega t}{2} \\ i\left(\cos\frac{\omega t}{2} - \sin\frac{\omega t}{2}\right) \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\cos\frac{\omega t}{2} + \sin\frac{\omega t}{2}\right)$$

$$|\langle +z|\Psi\rangle|^2 = \frac{1}{2}\left[\cos^2\frac{\omega t}{2} + \sin^2\frac{\omega t}{2} + 2\cos\frac{\omega t}{2}\sin\frac{\omega t}{2}\right]$$

$$= \frac{1}{2}\left[1 + \sin\omega_0 t\right]$$

Need $\sin\omega t = 1 \Rightarrow \omega t = \pi/2 \Rightarrow t = \frac{\pi}{2\omega}$

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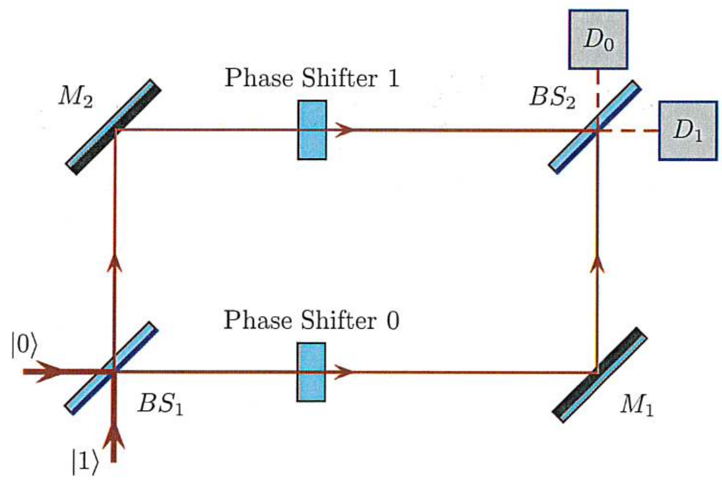
Question 2

Do **either** part a) or part b) for full credit.

a) A Mach-Zehnder interferometer consists of an arrangement of two beam splitters, BS_1 and BS_2, two mirrors, M_1 and M_2, and two detectors as illustrated. Note that the reflective side of B_1 is down and right and that of B_2 is up and left. Ignore the thickness of the glass in the beam-splitters and assume that they reflect 50% of the beam and transmit 50%.



The unitary operator for the beam splitters are

$$\hat{U}_{\text{BS1}} \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \hat{U}_{\text{BS2}} \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

That for passage between the beam splitters (this includes the phase shifters) is

$$\hat{U}_{\text{PS}} = \begin{pmatrix} e^{i\varphi_0} & 0 \\ 0 & e^{i\varphi_1} \end{pmatrix}.$$

Suppose that a single photon is in the state $|0\rangle$ prior to the first beam splitter. Determine the probabilit~~ies~~y with which it will emerge in the detector D_0. Determine the most general relationship between φ_0 and φ_1 such that the photon is equally likely to arrive in either detector.

Question 2 continued ...

Initially

$$|0\rangle \sim \begin{pmatrix}1\\0\end{pmatrix}$$

After BS1 state is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

After phase state is

$$\begin{pmatrix}e^{i\varphi_0} & 0\\0 & e^{i\varphi_1}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\varphi_0}\\e^{i\varphi_1}\end{pmatrix}$$

After second BS state is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\varphi_0}\\e^{i\varphi_1}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}e^{i\varphi_0}+e^{i\varphi_1}\\e^{i\varphi_1}-e^{i\varphi_0}\end{pmatrix} = \frac{1}{2}\left(e^{i\varphi_0}+e^{i\varphi_1}\right)|0\rangle$$

$$+\frac{1}{2}\left(e^{i\varphi_1}-e^{i\varphi_0}\right)|1\rangle \equiv |\Psi_f\rangle$$

$$\text{Prob}(D_0) = |\langle 0|\Psi_f\rangle|^2$$

$$= \left|\frac{1}{2}\left(e^{i\varphi_0}+e^{i\varphi_1}\right)\right|^2 = \frac{1}{4}\left(e^{-i\varphi_0}+e^{-i\varphi_1}\right)\left(e^{i\varphi_0}+e^{i\varphi_1}\right)$$

$$= \frac{1}{4}\Big(2+\underbrace{e^{i(\varphi_0-\varphi_1)}+e^{-i(\varphi_0-\varphi_1)}}_{2\cos(\varphi_0-\varphi_1)}\Big)$$

$$= \frac{1}{2}\left[1+\cos(\varphi_0-\varphi_1)\right]$$

We need this prob $=1/2$ $\Rightarrow$ $\cos(\varphi_0-\varphi_1)=0$

$$\Rightarrow \varphi_0-\varphi_1 = \frac{2n-1}{2}\pi$$

Question 2 continued ...

b) A state of light can be represented by

$$|\Psi\rangle = e^{-9/2} \sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n!}} |n\rangle$$

where $|n\rangle$ is a state such that a measurement of the number of photons yields exactly n. Suppose that the number of photons is measured. Determine the probability with which the measurement yields exactly zero photons **and** the expectation value of the number of photons.

$$\text{Prob}(0) = |\langle 0|\Psi\rangle|^2 = \left| e^{-9/2} \frac{3^0}{\sqrt{0!}} \right|^2$$

$$\text{Prob}(0) = e^{-9}$$

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Expectation value

$$\langle n\rangle = \sum n \, \text{Prob}(n)$$

$$\text{Prob}(n) = |\langle n|\Psi\rangle|^2 = \left(e^{-9/2} \frac{3^n}{\sqrt{n!}} \right)^2 = e^{-9} \frac{3^{2n}}{n!}$$

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$$\langle n\rangle = \sum_{n=0}^{\infty} e^{-9} n \frac{9^n}{n!} = e^{-9} \sum_{n=0}^{\infty} n \frac{9^n}{n!}$$

$$= e^{-9} \, 9 e^{9} = 9$$

$$\Rightarrow \langle n\rangle = 9$$

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Question 3

Particles of mass m are each in an infinite well with potential

$$V(x) = \begin{cases} 0 & 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

The (position) wavefunction corresponding to the n^{th} energy eigenstate, $|\phi_n\rangle$, with energy $E_n = n^2\pi^2\hbar^2/2mL^2$ is

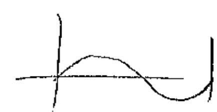
$$\phi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 \leqslant x \leqslant L \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that, at one instant, the position space wavefunction for the state of the particle is **restricted to half of the well** and is

$$\Psi(x) = \begin{cases} \sqrt{\frac{960}{L^5}}\, x \left(x - \frac{L}{2}\right) & 0 \leqslant x \leqslant \frac{L}{2} \\ 0 & \text{otherwise.} \end{cases}$$

This is normalized. Determine the probability with which an energy measurement will yield the outcome $E = 4\pi^2\hbar^2/2mL^2$.

The relevant energy is E_2. So we need



$$\text{Prob}(E_2) = |\langle\phi_2|\Psi\rangle|^2$$

$$\langle\phi_2|\Psi\rangle = \int_{-\infty}^{\infty} \phi_2(x)\Psi(x)\,dx$$

$$= \sqrt{\frac{960}{L^5}}\sqrt{\frac{2}{L}} \int_0^{L/2} \sin\left(\frac{2\pi x}{L}\right) x\left(x - L/2\right) dx$$

$$= \sqrt{\frac{1920}{L^6}} \int_0^{L/2} x\left(x - L/2\right) \sin\left(\frac{2\pi x}{L}\right) dx$$

Question 3 continued . . .

$$= \sqrt{\frac{1920}{L^6}} \left[\int_0^{L/2} x^2 \sin\left(\frac{2\pi x}{L}\right) dx - \frac{L}{2} \int_0^{L/2} x \sin\left(\frac{2\pi x}{L}\right) dx \right]$$

$$= \sqrt{\frac{1920}{L^6}} \left[\left\{ \cancel{\frac{2L}{2\pi} x^2 \sin\left(\frac{2\pi x}{L}\right)} + \left(\frac{2L^3}{8\pi^3} - \frac{x^2 L}{2\pi} \right) \cos\left(\frac{2\pi x}{L}\right) \right\}_0^{L/2} \right.$$

$$\left. - \frac{L}{2} \left\{ \cancel{\frac{L^2}{4\pi^2} \sin\left(\frac{2\pi x}{L}\right)} - \frac{xL}{2\pi} \cos\left(\frac{2\pi x}{L}\right) \right\}_0^{L/2} \right]$$

$$= \sqrt{\frac{1920}{L^6}} \left\{ \left(\frac{2L^3}{8\pi^3} - \frac{L^3}{8\pi} \right)(-1) + \frac{L^2}{4\pi} \left(\frac{L}{2} \cos\pi \right) \right\}$$

$$= \sqrt{1920} \left[\frac{1}{8\pi} - \frac{4}{8\pi^3} - \frac{1}{8\pi} \right] = -\sqrt{1920} \frac{1}{2\pi^3} = 0.707$$

$$\text{Prob} = \frac{1920}{4\pi^6} = \frac{480}{\pi^6}$$

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Question 4

Suppose that, at one instant, a particle in an infinite square well is known to be in one of the states

$$|\Psi_A\rangle = \frac{1}{\sqrt{2}} |\phi_2\rangle + \frac{i}{\sqrt{2}} |\phi_3\rangle$$
$$|\Psi_B\rangle = \frac{1}{\sqrt{2}} |\phi_2\rangle - \frac{i}{\sqrt{2}} |\phi_3\rangle$$

where $|\phi_n\rangle$ are energy eigenstates. Would measuring the energy of the particle enable you to determine whether the particle was in state A or state B with certainty, with some likelihood of success or neither? Explain your answer.

In general $\quad |\psi\rangle = \sum c_n |\phi_n\rangle$

$\text{Prob}(E_n) = |c_n|^2$

In this case, for both.

$$\left.\begin{aligned} \text{Prob}(E_2) &= \tfrac{1}{2} \\ \text{Prob}(E_3) &= \tfrac{1}{2} \end{aligned}\right\} \Rightarrow \text{Energy measurement will not succeed.}$$

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Question 5

A particle can move in one dimension. Its position space wavefunction is given by

$$\Psi(x,\cancel{t}) = \left(\frac{1}{\pi a^2}\right)^{1/4} e^{ip_0 x/\hbar}\, e^{-x^2/2a^2}$$

where p_0 has units of momentum.

a) Determine an expression for the momentum space wavefunction.

$$\tilde{\Psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{\pi a^2}\right)^{1/4} \int_{-\infty}^{\infty} e^{i(p_0-p)x/\hbar}\, e^{-x^2/2a^2}\, dx$$

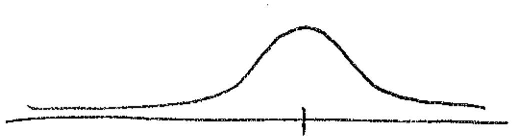
$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{\pi a^2}\right)^{1/4} \sqrt{\pi 2a^2}\; e^{(i(p_0-p)/\hbar)^2 a^2/2}$$

$$= \left(\frac{a^2}{\pi\hbar^2}\right)^{1/4} e^{-(p-p_0)^2 a^2/2\hbar^2}$$

b) Describe qualitatively what you would expect for the outcomes of a momentum measurement on a particle in this state.

The momentum probability density is $\tilde{P}(p) = |\tilde{\Psi}(p)|^2 = \sqrt{\frac{a^2}{\pi\hbar^2}}\, e^{-(p-p_0)^2 a^2/\hbar^2}$

This is a Gaussian centered at p_0. Measurement will give outcomes typically in the vicinity of p_0



p_0

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