Phys 321
Spring 2022

Quantum Theory I: Class Exam I

3 March 2022

Name: Solution **Total:** **/50**

Instructions

- There are 5 questions on 9 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Planck's constant	$h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $\quad \hbar = 1.05 \times 10^{-34}\,\mathrm{Js}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\,\mathrm{kg} = 511 \times 10^{3}\,\mathrm{eV/c^2}$
Mass of proton	$m_p = 1.673 \times 10^{-27}\,\mathrm{kg} = 938.3 \times 10^{6}\,\mathrm{eV/c^2}$
Mass of neutron	$m_n = 1.675 \times 10^{-27}\,\mathrm{kg} = 939.6 \times 10^{6}\,\mathrm{eV/c^2}$
Spherical coordinates	$\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$
Spin 1/2 state	$\lvert +\hat{\boldsymbol{n}}\rangle = \cos(\theta/2)\lvert +\hat{\boldsymbol{z}}\rangle + e^{i\phi}\sin(\theta/2)\lvert -\hat{\boldsymbol{z}}\rangle$
Spin 1/2 state	$\lvert -\hat{\boldsymbol{n}}\rangle = \sin(\theta/2)\lvert +\hat{\boldsymbol{z}}\rangle - e^{i\phi}\cos(\theta/2)\lvert -\hat{\boldsymbol{z}}\rangle$
Euler relation	$e^{i\alpha} = \cos\alpha + i\sin\alpha$
Spin observables	$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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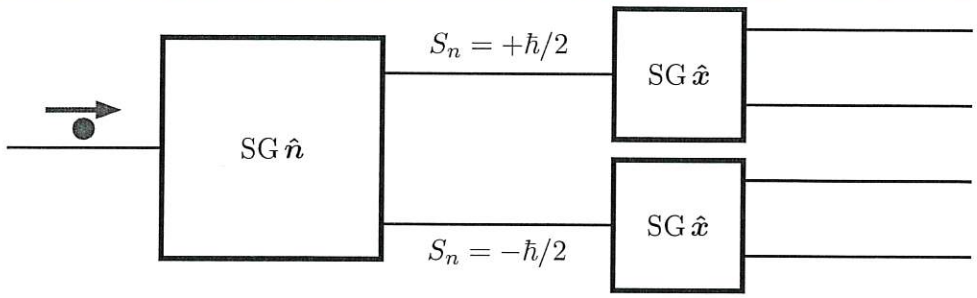
Question 1

A spin-1/2 particle is subjected to an SG $\hat{\boldsymbol{n}}$ apparatus, where $\hat{\mathbf{n}} = -\frac{1}{\sqrt{2}}\hat{\mathbf{x}} + \frac{1}{\sqrt{2}}\hat{\mathbf{z}}$, and then subsequently an SG $\hat{\boldsymbol{x}}$ apparatus.



a) Suppose that the particle emerges from the first measurement in the beam for which $S_n = +\hbar/2$. Determine an expression for the ket, in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$, that represents the state of the particle after it emerges from the SG $\hat{\boldsymbol{n}}$ apparatus.

The state will be $|+\hat{n}\rangle$ here $\phi = \pi$ $\theta = \pi/4$

$$|+\hat{n}\rangle = \cos\left(\tfrac{\theta}{2}\right)|+\hat{z}\rangle + e^{i\phi}\sin\left(\tfrac{\theta}{2}\right)|-\hat{z}\rangle$$

$$= \cos\left(\tfrac{\pi}{8}\right)|+\hat{z}\rangle + e^{i\pi}\sin\left(\tfrac{\pi}{8}\right)|-\hat{z}\rangle$$

$$= \cos\left(\tfrac{\pi}{8}\right)|+\hat{z}\rangle - \sin\pi/8\,|-\hat{z}\rangle$$

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b) The particle is subsequently subjected to an SG $\hat{\boldsymbol{x}}$ apparatus. List the measurement outcomes and the probabilities with which they occur.

Outcome: $S_x = +\hbar/2$ $\quad$ $\text{Prob}(S_x = +\hbar/2) = |\langle +\hat{x}|+\hat{n}\rangle|^2$

then $|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$

$$\langle +\hat{x}|+\hat{n}\rangle = \frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix}\cos\pi/8 \\ -\sin\pi/8\end{pmatrix} = \frac{1}{\sqrt{2}}(\cos\pi/8 - \sin\pi/8)$$

$$= (\sin\pi/4\cos\pi/8 - \cos\pi/4\sin\pi/8)$$

$$= \sin\left(\tfrac{\pi}{4} - \pi/8\right) = \sin\pi/8$$

Question 1 continued ...

So $\quad \text{Prob}(S_x = +\hbar/2) = \sin^2 \pi/8 = 0.15$

Outcome $-\hbar/2$ $\qquad \text{Prob}(S_x = -\hbar/2) = |\langle -\hat{x}|+\hat{n}\rangle|^2$

$$|-\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{1}{\sqrt{2}}|-\hat{z}\rangle$$

$$\langle -\hat{x}|+\hat{n}\rangle = \frac{1}{\sqrt{2}}(1 \; -1)\begin{pmatrix} \cos \pi/8 \\ -\sin \pi/8 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\cos \pi/8 + \sin \pi/8\right)$$

$$= \cos \pi/4 \cos \pi/8 + \sin \pi/4 \sin \pi/8$$

$$= \cos \pi/8$$

$$\text{Prob}(S_x = -\hbar/2) = \cos^2 \pi/8 = 0.85$$

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c) Suppose that the pair of measurements is regarded as a single measurement, yielding a pair of outcomes: one for S_n and one for S_x. Describe whether this pair of measurements is repeatable, i.e. whether a particle that yields a particular pair of outcomes and is then subjected the same measurement again will yield the same pair of outcomes with certainty.

No, suppose that one obtains $S_n = +\hbar/2$ $\quad S_x = +\hbar/2$. Then the state prior to repeat will be $|+\hat{x}\rangle$. But this could yield $S_n = -\hbar/2$, $S_x = +\hbar/2$. Thus the same outcomes won't occur with certainty

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Question 2

Determine matrix representations for the measurement operators associated with each of the two outcomes of an S_y measurement.

$S_y = +\hbar/2 \rightsquigarrow \hat{P}_{+y} = |+\hat{y}\rangle\langle+\hat{y}|$

$S_y = -\hbar/2 \rightsquigarrow P_{-y} = |-\hat{y}\rangle\langle-\hat{y}|$

$|+\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$|-\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$\hat{P}_{+y} \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}(1\ -i)\frac{1}{\sqrt{2}} = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$

$\hat{P}_{-y} \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\frac{1}{\sqrt{2}}(1\ i) = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$

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Question 3

A large collection of particles are all known to be in the state

$$|\Psi_i\rangle = \frac{4}{5}\,|+\hat{z}\rangle + \frac{3i}{5}\,|-\hat{z}\rangle \quad \rightsquigarrow \quad \frac{1}{5}\begin{pmatrix} 4 \\ 3i \end{pmatrix}$$

at an initial instant.

a) A quarter of these particles are subjected to a measurement of S_z. Determine the expected value of the average measurement outcome, $\langle S_z \rangle$.

$$\langle S_z \rangle = \langle \Psi_i | \hat{S}_z | \Psi_i \rangle$$

$$= \frac{1}{5}\begin{pmatrix} 4 & -3i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\,\frac{1}{5}\begin{pmatrix} 4 \\ 3i \end{pmatrix}$$

$$= \frac{\hbar}{50}\begin{pmatrix} 4 & -3i \end{pmatrix}\begin{pmatrix} 4 \\ -3i \end{pmatrix} = \frac{\hbar}{50}\,(16-9) \quad \frac{\hbar}{2}\,\frac{7}{25}$$

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b) Another quarter of these particles are subjected to a measurement of S_x. Determine the expected value of the average measurement outcome, $\langle S_x \rangle$.

$$\langle S_x \rangle = \frac{1}{5}\begin{pmatrix} 4 & -3i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 4 \\ 3i \end{pmatrix} = \frac{\hbar}{50}\begin{pmatrix} 4 & -3i \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$= 0$$

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Question 3 continued ...

c) The remaining half of the particles are subjected to an evolution described by evolution operator

$$\hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Determine the state of the particles after the evolution.

$$|\Psi_f\rangle = \frac{1}{5}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 4 \\ 3i \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

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d) If the remaining particles were subdivided and S_z measured on one half and S_x measured on the other would the resulting expected values of the average measurement outcomes change? Explain your answer.

$$\langle S_z \rangle = \frac{1}{5}(4\ -3)\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \frac{\hbar}{50}(4\ -3)\begin{pmatrix} 4 \\ +3 \end{pmatrix} = \frac{\hbar}{2}\frac{7}{25}$$

SAME

$$\langle S_x \rangle = \frac{1}{5}(4\ -3)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \frac{\hbar}{50}(4\ -3)\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \frac{\hbar}{2}\left(\frac{-24}{25}\right)$$

different! 4

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Question 4

Consider the two states

$$|\psi_1\rangle = \frac{1}{2}|+\hat{z}\rangle + \frac{\sqrt{3}}{2}|-\hat{z}\rangle$$
$$|\psi_2\rangle = \frac{1}{2}|+\hat{z}\rangle - \frac{\sqrt{3}}{2}|-\hat{z}\rangle$$

Can each of these states possibly be associated with each of the outcomes $S_n = +\hbar/2$ and $S_n = -\hbar/2$ for a single spin component measurement? Explain your answer.

For this to be true they must be orthogonal:

$$\langle\psi_1|\psi_2\rangle = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1}{4}(1-3) = -\frac{1}{2} \neq 0$$

They are <u>not</u>, so they cannot be associated with a single S_n measurement.

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Question 5

An scientist observes the evolution of a spin-1/2 system by observing what happens to particles in the states $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$. Based on these, the scientist proposes one of the two following evolution operators

$$\hat{U}_1 = |+\hat{x}\rangle \langle +\hat{z}| + |-\hat{x}\rangle \langle -\hat{z}| \qquad \text{or} \qquad \hat{U}_2 = |+\hat{x}\rangle \langle +\hat{z}| + |-\hat{y}\rangle \langle -\hat{z}|$$

Determine the matrices that represent each of these and check which is a possible evolution operator.

$$\hat{U}_1 \sim \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\0 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$\hat{U}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\0 & -i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -i\end{pmatrix}$$

Need $\hat{U}_i^{+}\hat{U}_i = \hat{I}$

$$\hat{U}_1^{+}\hat{U}_1 = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad ✓✓$$

$$\hat{U}_2^{+}\hat{U}_2 = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & i\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 1-i\\1+i & 2\end{pmatrix} \neq \hat{I} \qquad XX$$

Only U_1 works!

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