Diagnostic Test: Please do the "Post" test.

Final Exam: Covers Entire Semester.

Review 2007, 2020 } All questions

Thurs: Read 4.1

Particle in a central potential: energy eigenstates

A particle in a central potential can be described by Hamiltonian

$$\hat{H} = \frac{1}{2M}\hat{P}_r^2 + \frac{1}{2M\hat{r}^2}\hat{\vec{L}}^2 + V(\hat{r})$$

where $V(\hat{r})$ is the potential for the system. We can find simultaneous eigenstates of $\hat{H}, \hat{\vec{L}}^2, \hat{L}_z$ and we denote these as

$$|\phi\rangle = |E, l, m\rangle$$

where

$$\hat{H}|E,l,m\rangle = E|E,l,m\rangle$$

$$\hat{\vec{L}}^2|E,l,m\rangle = \hbar^2 l(l+1)|E,l,m\rangle$$

$$\hat{L}_z|E,l,m\rangle = \hbar m|E,l,m\rangle$$

Then $l = 0, 1, 2, 3, \ldots$

$$m = -l, -l+1, \ldots l-1, l.$$

Immediately we have

$$\hat{H}|E,l,m\rangle = \frac{1}{2m}\hat{p}_r^{\,2}|E,l,m\rangle + \frac{1}{2m\hat{r}^2}\underbrace{\hat{\vec{L}}^2|E,l,m\rangle}_{\hbar^2 l(l+1)|E,l,m\rangle} + V(\hat{r})|E,l,m\rangle$$

$$\hat{H}|E,l,m\rangle = \frac{1}{2m}\hat{p}_r^{\,2}|E,l,m\rangle + \frac{\hbar^2}{2m\hat{r}^2}l(l+1)|E,l,m\rangle + V(\hat{r})|E,l,m\rangle = E|E,l,m\rangle$$

We can convert this into an equation for the radial wavefunction

$$|E,l,m\rangle = R_{El}(r)\,Y_{lm}(\theta,\phi)$$

Then we get:

$$\boxed{-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)^2 R_{El} + \frac{\hbar^2 l(l+1)}{2mr^2}R_{El} + V(r)R_{El} = E R_{El}}$$

The radial wavefunction can clearly depend on both E and l. Solving this will produce energy eigenvalues E.

Hydrogen atom

For the hydrogen atom

$$V(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

where e = electron charge
ϵ_0 = permittivity of free space.

Thus:

$$\frac{-\hbar^2}{2M}\left[\frac{d^2R_{E\ell}}{dr^2}+\frac{2}{r}\frac{dR_{E\ell}}{dr}-\cancel{\frac{R_{E\ell}}{r^2}}+\cancel{\frac{R_{E\ell}}{r^2}}\right]+\frac{\hbar^2\ell(\ell+1)}{2Mr^2}R_{E\ell}-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}R_{E\ell}=ER_{E\ell}.$$

$$\Rightarrow \frac{d^2R_{E\ell}}{dr^2}+\frac{2}{r}\frac{dR_{E\ell}}{dr}+\frac{2M}{\hbar^2}\left[E+\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}-\frac{\hbar^2\ell(\ell+1)}{2Mr^2}\right]R_{E\ell}=0$$

We first rescale the variable using

$$\rho := r/a$$

where a has units of length. So

$$r=\rho a \quad\Rightarrow\quad \frac{d}{dr}=\frac{d\rho}{dr}\frac{d}{d\rho}=\frac{1}{a}\frac{d}{d\rho}.$$

Thus:

$$\frac{1}{a^2}\frac{d^2R_{E\ell}}{d\rho^2}+\frac{1}{a^2}\frac{2}{\rho}\frac{dR_{E\ell}}{d\rho}+\frac{2M}{\hbar^2}\left[E+\frac{e^2}{4\pi\epsilon_0}\frac{1}{a}\frac{1}{\rho}-\frac{\hbar^2\ell(\ell+1)}{2Ma^2}\frac{1}{\rho^2}\right]R_{E\ell}=0$$

$$\Rightarrow \frac{d^2R_{E\ell}}{d\rho^2}+\frac{2}{\rho}\frac{dR_{E\ell}}{d\rho}+\left[\frac{2ME}{\hbar^2}a^2+\frac{me^2}{2\pi\hbar^2\epsilon_0}\frac{a}{\rho}-\frac{\ell(\ell+1)}{\rho^2}\right]R_{E\ell}=0$$

Choose

$$a=\frac{4\pi\hbar^2\epsilon_0}{me^2} \quad\Rightarrow\quad \rho=r/a=\frac{me^2}{4\pi\hbar^2\epsilon_0}r$$

$$\Rightarrow \boxed{\frac{d^2R_{E\ell}}{d\rho^2}+\frac{2}{\rho}\frac{dR_{E\ell}}{d\rho}+\left[\frac{2mE}{\hbar^2}a^2+\frac{2}{\rho}-\frac{\ell(\ell+1)}{\rho^2}\right]R_{E\ell}(\rho)=0}$$

There is a systematic approach to solving these. We can illustrate some cases.

<u>Example:</u> $R_{E\ell} = C \quad e^{-\gamma\rho}$

$\quad\quad\quad\quad\quad\quad\quad$ $\nwarrow$ constant.

$$\Rightarrow \quad \gamma^2 R_{E\ell} - \frac{2\gamma}{\rho} R_{E\ell} + \frac{2mE}{\hbar^2} a^2 R_{E\ell} + \frac{2R_{E\ell}}{\rho} - \frac{\ell(\ell+1)}{\rho^2} R_{E\ell} = 0$$

This can only be satisfied for all ρ if:

$\frac{1}{\rho^2}$ term $\quad \ell(\ell+1) = 0 \quad \Rightarrow \quad \ell = 0$

$\frac{1}{\rho}$ term $\quad -2\gamma + 2 = 0 \quad \Rightarrow \quad \gamma = 1$

$\frac{1}{\rho^0}$ term $\quad \gamma^2 + \frac{2mEa^2}{\hbar^2} = 0 \quad \Rightarrow \quad E = \frac{-\gamma^2\hbar^2}{2ma^2}$

Thus we get a solution with:

$\ell = 0$

$$E = -\frac{\hbar^2}{2ma^2} = -\frac{\hbar^2}{2m}\frac{m^2 e^4}{16\pi^2\hbar^4\epsilon_0^2}$$

$$E = -\frac{1}{2}\frac{m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \qquad R_{E\ell}(\rho) = Ce^{-\rho}$$

This is actually the ground state energy.

For a systematic approach see Ch 8

Multiple distinguishable particles

It is possible to consider situations involving multiple quantum systems. We consider <u>two distinguishable spin-½ particles.</u> We need to extend the framework of quantum theory to describe

* measurements
* states
* evolution

of such pairs of particles.

Particle A $\rightsquigarrow$ $\boxed{\text{meas } S_n}$??

Particle B $\rightsquigarrow$ $\boxed{\text{meas } S_n}$??

A starting point would be to assemble products of states, of the form

$$|\Psi_1\rangle_A \, |\Psi_2\rangle_B$$

where the left ket refers to particle A and the right to particle B. Examples and their interpretations are:

1) $|+z\rangle_A |+z\rangle_B \Rightarrow$ measure S_z on A $\rightsquigarrow$ get $+\hbar/2$ with certainty
AND
measure S_z on B $\rightsquigarrow$ " $+\hbar/2$ with certainty

2) $|+\hat{z}\rangle_A |-z\rangle_B \Rightarrow$ measure S_z on A $\Rightarrow +\hbar/2$ with certainty
measure S_z on B $\Rightarrow -\hbar/2$ " "

3) $|-\hat{z}\rangle_A |+\hat{z}\rangle_B \Rightarrow$ measure S_z on A $\Rightarrow -\hbar/2$ with certainty
measure S_z on B $\Rightarrow +\hbar/2$ with certainty

We can create bra vectors using the notation

$|\Phi_1\rangle_A|\Phi_2\rangle_B$ ket $\Rightarrow$ associated bra is $\langle\Phi_1|_A\langle\Phi_2|_B$

Then the inner product has a core definition

$$(\langle\Phi_1|_A\langle\Phi_2|_B)(|\Psi_1\rangle_A|\Psi_2\rangle_B) = \underbrace{(\langle\Phi_1|\Psi_1\rangle)(\langle\Phi_2|\Psi_2\rangle)}_{\text{product of two complex numbers}}$$

Now we can describe measurement statistics via:

Let A be an observable quantity measured on particle A
Let B " " " " " " " B

List outcomes and states for each measurement.

Measurement A

outcome	state
a_1	$|\phi_1\rangle_A$
a_2	$|\phi_2\rangle_A$

Measurement B

outcome	state
b_1	$|\chi_1\rangle_B$
b_2	$|\chi_2\rangle_B$

List joint outcomes and states.

outcome A	outcome B	state
a_1	b_1	$|\phi_1\rangle_A|\chi_1\rangle_B$
a_1	b_2	$|\phi_1\rangle_A|\chi_2\rangle_B$
a_2	b_1	$|\phi_2\rangle_A|\chi_1\rangle_B$
a_2	b_2	$|\phi_2\rangle_A|\chi_2\rangle_B$

$\Rightarrow$ Calculate probabilities

Prob (a_1 AND b_1)

$$= |\langle\phi_1|_A\langle\chi_1|_B|\underbrace{\Psi}_{\text{for AB}}\rangle|^2$$

etc...

1 Measurements on pairs of spin-1/2 particles

Consider two spin-1/2 particles in the state

$$|+\hat{x}\rangle_A |-\hat{x}\rangle_B .$$

a) Suppose that S_z is measured for each particle. List all possible outcomes and their probabilities.

b) Suppose that S_x is measured for each particle. List all possible outcomes and their probabilities.

c) Suppose that S_x is measured for particle A and S_z . List all possible outcomes and their probabilities.

Answer: a)

Particle A	Particle B	Prob
$+\hbar/2$	$+\hbar/2$	$\lvert(\langle +z\lvert\langle +z\rvert)(\lvert +x\rangle \lvert +x\rangle)\rvert^2 = \frac{1}{4}$
$+\hbar/2$	$-\hbar/2$	$\frac{1}{4}$
$-\hbar/2$	$+\hbar/2$	$\frac{1}{4}$
$-\hbar/2$	$-\hbar/2$	$\frac{1}{4}$

For the first row

$$\langle +z|\langle +z|\,|+\hat{x}\rangle|+\hat{x}\rangle = \langle +z|+\hat{x}\rangle\langle +\hat{z}|+\hat{x}\rangle$$

Then $|+x\rangle = \frac{1}{\sqrt{2}}\left(|+\hat{z}\rangle + |-\hat{z}\rangle\right) \Rightarrow \langle +z|+\hat{x}\rangle = \frac{1}{\sqrt{2}}$. Thus

$$\langle +\hat{z}|\langle +z|\,|+\hat{x}\rangle|+\hat{x}\rangle = \frac{1}{2}$$

This gives the probability. The others follow in the same way.

b) $\text{Prob}\,(S_x = +\hbar/2,\ S_x = +\hbar/2) = \left| \langle +x | \langle +x| \; |+x\rangle \; |-\hat{x}\rangle \right|^2$

$$= \Big| \underbrace{\overset{A}{\langle +x | +\hat{x}\rangle}}_{1} \; \underbrace{\overset{B}{\langle +x | -\hat{x}\rangle}}_{0} \Big|^2 = 0$$

$\text{Prob}\,(S_x = +\hbar/2,\ S_x = -\hbar/2) = \left| \langle +x | +x\rangle \langle -x | -\hat{x}\rangle \right|^2 = 1$

etc... So we get

Particle A S_x	Particle B S_x	Probability
$+\hbar/2$	$+\hbar/2$	0
$+\hbar/2$	$-\hbar/2$	1
$-\hbar/2$	$+\hbar/2$	0
$-\hbar/2$	$-\hbar/2$	0

c) $\text{Prob}\,(S_x = +\hbar/2,\ S_z = +\hbar/2) = \left| \langle +x | \langle +z| \; |+x\rangle |+x\rangle \right|^2$

$$= \Big| \underbrace{\langle +x | +x\rangle}_{1} \underbrace{\langle -z | +x\rangle}_{\frac{1}{\sqrt{2}}} \Big|^2 = \frac{1}{2}$$

$\text{Prob}\,(S_x = +\hbar/2,\ S_z = -\hbar/2) = \left| \langle +x | \langle -z | \, |+x\rangle |+x\rangle \right|^2$

$$= \Big| \underbrace{\langle +x | +x\rangle}_{1} \underbrace{\langle -z | +x\rangle}_{\frac{1}{\sqrt{2}}} \Big|^2 = \frac{1}{2}$$

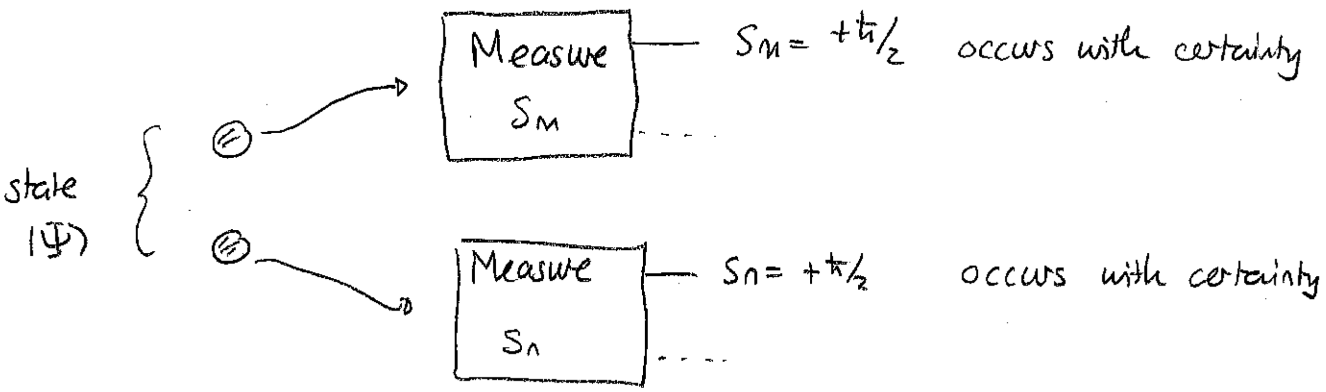
The others are zero

A S_x	B S_z	Prob
+	+	1/2
+	−	1/2
−	+	0
−	−	0

These are all examples of product states. For any pair of spin-$1/2$ particles such a product state has the form

$$\underbrace{|\Psi\rangle}_{\substack{\text{represents}\\ \text{both particles}}} = \underbrace{|+\hat{m}\rangle_A}_{\substack{\text{represents}\\ \text{system A}}} \underbrace{|+\hat{n}\rangle_B}_{\substack{\text{represents}\\ \text{system B.}}}$$

where $\hat{m}$ and $\hat{n}$ are some directions. This has the interpretation



Thus

For a product state there is a measurement for A and a measurement for B such that one pair of outcomes will occur with certainty.

Superpositions of product states.

We can create superpositions of product states using tensor product multiplication. For example

$$|+x\rangle_A = \frac{1}{\sqrt{2}}|+\hat{z}\rangle_A + \frac{1}{\sqrt{2}}|-\hat{z}\rangle_A$$

$$|+x\rangle_B = \frac{1}{\sqrt{2}}|+\hat{z}\rangle_A + \frac{1}{\sqrt{2}}|-\hat{z}\rangle_A.$$

Then

$$|+x\rangle_A |+x\rangle_B = \left(\frac{1}{\sqrt{2}} |+z\rangle_A + \frac{1}{\sqrt{2}} |-z\rangle_A\right)\left(\frac{1}{\sqrt{2}} |+z\rangle_B + \frac{1}{\sqrt{2}} |-z\rangle_B\right)$$

$$= \frac{1}{2}\left[|+z\rangle_A |+z\rangle_B + |+z\rangle_A |-z\rangle_B + |-z\rangle_A |+z\rangle_B + |-z\rangle_A |-z\rangle_B\right]$$

This is a superposition of products. Such superpositions occur frequently and can be used to do calculations.

In general we will find superpositions of the form:

$$|\Psi\rangle = a_0 |+z\rangle|+z\rangle + a_1 |+z\rangle|-z\rangle + a_2 |-z\rangle|+z\rangle + a_3 |-z\rangle|-z\rangle$$

where a_0, a_1, a_2, a_3 are complex numbers

2 Superpositions of product states

Consider the states

$$|\Psi_1\rangle = |+\hat{y}\rangle_A |+\hat{y}\rangle_B$$
$$|\Psi_2\rangle = |+\hat{y}\rangle_A |-\hat{y}\rangle_B$$

a) Express each as a superposition of $|+\hat{z}\rangle_A |+\hat{z}\rangle_B, |+\hat{z}\rangle_A |-\hat{z}\rangle_B, \ldots$

b) Use the superpositions to show that these states are orthonormal.

Answer.

$$|+\hat{y}\rangle = \frac{1}{\sqrt{2}}\left[|+\hat{z}\rangle + i|-\hat{z}\rangle\right]$$

$$|-\hat{y}\rangle = \frac{1}{\sqrt{2}}\ |+\hat{z}\rangle - i\,|-\hat{z}\rangle]$$

a) $$|\Psi_1\rangle = \frac{1}{2}\left[|+z\rangle|+z\rangle + i|+z\rangle|-z\rangle + i\,|-z\rangle|+z\rangle - |-z\rangle|-\hat{z}\rangle\right]$$

$$|\Psi_2\rangle = \frac{1}{2}\left[|+z\rangle|+z\rangle - i\,|+z\rangle|-z\rangle + i\,|-z\rangle|+z\rangle - |-z\rangle|-z\rangle\right].$$

b) $$\langle\Psi_1| = \frac{1}{2}\left[\langle+z|\langle+z| - i\langle+z|\langle-z| - i\langle-z|\langle+z| - \langle-z|\langle-z\right]$$

$$\langle\Psi_2| = \frac{1}{2}\left[\langle+z|\langle+z| + i\langle+z|\langle-z| - i\langle-z|\langle+z| - \langle-z|\langle-z|\right].$$

So

$$\langle\Psi_1|\Psi_1\rangle = \frac{1}{4} + \frac{i}{2}\left(\frac{-i}{2}\right) + \frac{i}{2}\left(\frac{-i}{2}\right) + \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right) = 1$$

$$\langle\Psi_2|\Psi_2\rangle = \frac{1}{2}\frac{1}{2} + \frac{i}{2}\left(\frac{-i}{2}\right) + \frac{i}{2}\left(\frac{-i}{2}\right) + \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right) = 1$$

$$\langle\Psi_2|\Psi_1\rangle = \frac{1}{2}\frac{1}{2} + \frac{i}{2}\left(\frac{i}{2}\right) + \frac{i}{2}\left(\frac{i}{2}\right) + \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right) = 0$$

They are orthonormal.