Phys 321 2023

Lecture 26

Fri: HW 5pm

Tues: Read 8.1 → 8.2, 2.7

HW 5pm

Diagnostic Test:

## Angular momentum

We found that the possible eigenstate structure for angular momentum states is.

States are denoted

$|j,m\rangle$

where $j$ labels value for $\vec{J}^2$

and $m$ labels value for $J_z$

Satisfy

$$\hat{\vec{J}}^2 |j,m\rangle = \hbar j(j+1)|j,m\rangle$$

$$\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$$

Possible values are:

Given $j = 0\ \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$

then

$$m = -j, -j+1, \ldots j-1, j$$

Information about the x and y components can be determined by

$$\hat{J}_\pm = \hat{J}_x \pm i \hat{J}_y$$

$$\hat{J}_\pm |j,m\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)}\ |j, m\pm 1\rangle$$

We usually distinguish between spin and orbital angular momentum by replacing

| For orbital | For spin |
|---|---|
| $\vec{J} \to \vec{L}$ | $\vec{J} \to \vec{S}$ |
| $j \to l$ | $j \to j$ |

This still leaves the question of which values for $j$ are allowed for each. There is a clear answer for orbital angular momentum

## Orbital angular momentum

Orbital angular momentum is related to spatial co-ordinates. We have seen that in terms of spherical co-ordinates:

$$\hat{\vec{L}}^2 \rightsquigarrow -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\hat{L}_z \rightsquigarrow -i\hbar\frac{\partial}{\partial\phi}$$

We can then seek wavefunctions for each possible set of values of $l, m$. So we would represent

$$|l,m\rangle \to \Psi_{lm}(r,\theta,\phi)$$

and we can split this into radial and angular functions

$$\Psi_{lm}(r,\theta,\phi) = R_{lm}(r)\,Y_{lm}(\theta,\phi)$$

Here these are arranged to satisfy normalization requirements via:

$$\boxed{\begin{aligned} &\int_0^\infty r^2 |R_{\ell m}(r)|^2 dr = 1 \\ &\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \; |Y_{\ell m}(\theta,\phi)|^2 = 1 \end{aligned}}$$

Then in eigenvalue equations the radial part factors and

$$\hat{L}^2 |\ell,m\rangle = \hbar^2 \ell(\ell+1) |\ell,m\rangle$$

$$\leadsto \; -\hbar^2 \left[ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial Y_{\ell m}}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{\ell m}}{\partial\phi^2} \right] = \hbar^2 \ell(\ell+1) Y_{\ell m}(\theta,\phi)$$

$$\hat{L}_z |\ell,m\rangle = \hbar m |\ell,m\rangle$$

$$\leadsto \; -i\hbar \frac{\partial}{\partial\phi} Y_{\ell m}(\theta,\phi) = \hbar m \, Y_{\ell,m}(\theta,\phi)$$

We will be able to use these to narrow down values of $\ell$ (from the possible $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$) that apply to orbital angular momentum.

## 1 Orbital angular momentum states

Show that the state

$$Y_{1,1}(\theta,\phi) := \sqrt{\frac{3}{8\pi}}\,\sin(\theta)\,e^{i\phi}$$

satisfies the orbital angular momentum equations and identify the state $|l,m\rangle$ to which it corresponds.

Answer: We need to consider

$\hat{L}^2 |l,m\rangle$ $\rightsquigarrow$ identify $l$

and

$\hat{L}_z |l,m\rangle$ $\rightsquigarrow$ identify $m$.

In terms of wavefunctions

$$\hat{L}^2 |l,m\rangle \rightsquigarrow -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\sqrt{\frac{3}{8\pi}}e^{i\phi}\right) + \frac{1}{\sin^2\theta}(-1)\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right]$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\sqrt{\frac{3}{8\pi}}e^{i\phi}\underbrace{\left(\cos^2\theta - \sin^2\theta\right)}_{1-2\sin^2\theta} - \frac{1}{\sin^{\cancel{2}}\theta}\sqrt{\frac{3}{8\pi}}\cancel{\sin\theta}\, e^{i\phi}\right]$$

$$= -\hbar^2\left[\sqrt{\frac{3}{8\pi}}e^{i\phi}\left(\cancel{\frac{1}{\sin\theta}} - 2\sin\theta\right) - \sqrt{\frac{3}{8\pi}}e^{i\phi}\cancel{\frac{1}{\sin\theta}}\right]$$

$$= 2\hbar^2\underbrace{\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}}_{Y_1(\theta,\phi)}$$

Thus $Y(\theta,\phi)$ is clearly an eigenstate of $\hat{L}^2$ with eigenvalue $\hbar^2 2$.
Possible eigenvalues of $\hat{L}^2$ are $\hbar^2 l(l+1)$.

Thus

$$\hbar^2 2 = \hbar^2 l(l+1)$$

$$\Rightarrow \quad l^2 + l - 2 = 0 \quad \Rightarrow \quad (l-1)(l+2) = 0 \quad \Rightarrow \quad \begin{matrix} l = 1 \\ l = -2 \end{matrix}$$

Only positive $l$ are possible. So $l=1$

Then

$$\hat{L}_z |l,m\rangle \sim -i\hbar \frac{\partial}{\partial\phi} Y(\theta,\phi) = -i\hbar \frac{\partial}{\partial\phi}\left[\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}\right]$$

$$= -i\hbar \sqrt{\frac{3}{8\pi}} \sin\theta \,(i e^{i\phi})$$

$$= \hbar \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}$$

$$= \hbar\, Y(\theta,\phi)$$

So this is an eigenstate with eigenvalue $\hbar$. Possible eigenvalues of $\hat{L}_z$ are $\hbar m$. Thus $m=1$

Thus the state is $|1,1\rangle$ ∎

The states $Y_{\ell m}(\theta,\phi)$ that are simultaneously eigenstates of $\hat{L}^2$ and $\hat{L}_z$ are called spherical harmonics. In general they have a separable form

$$Y_{\ell m}(\theta,\phi) = f_{\ell m}(\theta)\, g_{\ell m}(\phi)$$

Then

$$\hat{L}_z Y_{\ell m}(\theta,\phi) = m\hbar\, Y_{\ell m}$$

$$\Rightarrow \quad -i\hbar \frac{\partial}{\partial\phi} f_{\ell m}(\theta)\, g_{\ell m}(\phi) = m\hbar\, f_{\ell m}(\theta)\, g_{\ell m}(\phi)$$

$$\Rightarrow \quad -i\hbar\, \cancel{f_{\ell m}} \frac{d g_{\ell m}}{d\phi} = m\hbar\, \cancel{f_{\ell m}}\, g_{\ell m}$$

$$\Rightarrow \quad -i \frac{d g_{\ell m}}{d\phi} = m\, g_{\ell m}$$

$$\Rightarrow \quad \frac{d g_{\ell m}}{d\phi} = i m\, g_{\ell m}.$$

Then the solutions to these are:

$$g_{\ell m}(\phi) = \text{const} \times e^{im\phi}$$

We now require

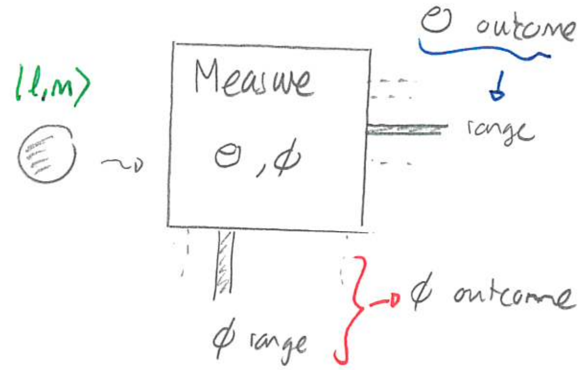
$$g_{\ell m}(0) = g_{\ell m}(2\pi) \quad \Rightarrow \quad 1 = e^{im2\pi}$$

This is only possible if $m$ is an integer. $\Rightarrow$ $j$ is an integer. Thus

$$\boxed{\text{For orbital angular momentum } j = \ell = 1, 2, 3, 4, \ldots \text{ and possible values of } m \text{ are integers.}}$$

# Spherical harmonics and angular probability densities.

Consider a situation where a particle is in the orbital angular momentum state $|\ell, m\rangle$. We could measure it's angular position. The probability with which the outcome will be in certain ranges, regardless of $r$ is




$$\text{Prob}\left[\theta_a \le \theta \le \theta_b \ \text{ AND } \ \phi_a \le \phi \le \phi_b\right]$$

$$= \int_0^\infty dr \int_{\theta_a}^{\theta_b} d\theta \int_{\phi_a}^{\phi_b} d\phi \ r^2 \sin\theta \ \Psi_{\ell m}(r,\theta,\phi)$$

$$= \underbrace{\int_0^\infty r^2 |R_{\ell m}(r)|^2 dr}_{=1} \int_{\theta_a}^{\theta_b} d\theta \int_{\phi_a}^{\phi_b} d\phi \ \sin\theta \ |Y_{\ell m}(\theta,\phi)|^2$$

Thus the angular probability density is

$$\boxed{P(\theta,\phi) = \sin\theta \ |Y_{\ell m}(\theta,\phi)|^2}$$

Similarly if we have superpositions, the probability will be determined by the relevant superpositions of spherical harmonics. So we need spherical harmonics to predict the outcomes of angular position measurements.

Thus we need a method to generate spherical harmonics. The strategy involves raising and lowering operators, expressed in terms of derivatives.

We use:

$$\hat{L}_x \rightsquigarrow i\hbar \left[\sin\phi \frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta} \cos\phi \frac{\partial}{\partial\phi}\right]$$

$$\hat{L}_y \rightsquigarrow i\hbar \left[-\cos\phi \frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta} \sin\phi \frac{\partial}{\partial\phi}\right]$$

$\hookrightarrow$

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y \rightsquigarrow \hbar e^{i\phi}\left[\frac{\partial}{\partial\theta} + i \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\phi}\right]$$

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y \rightsquigarrow \hbar e^{-i\phi}\left[-\frac{\partial}{\partial\theta} + i \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\phi}\right]$$

$\hookrightarrow$ For a given $l$ we know

$$\hat{L}_- |l, -l\rangle = 0$$

$$\Rightarrow \hbar e^{-i\phi}\left[-\frac{\partial}{\partial\theta} + i \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\phi}\right] Y_{l-l}(\theta,\phi) = 0$$

This gives $Y_{l-l}(\theta,\phi)$

get $Y_{l-l}(\theta,\phi)$ lowest in ladder

$\hookrightarrow$ Then

$$\hat{L}_+ |l, -l\rangle = \hbar \sqrt{l(l+1) - (-l)(-l+1)}\ |l, -l+1\rangle$$

and this will give $Y_{l,-l+1}(\theta,\phi)$

generates second lowest in ladder

$\vdots$

$\hookrightarrow$ Iterating the process generates all states

## 2 Constructing spherical harmonics

Consider the various angular momentum states $|1, m\rangle$ for $m = -1, 0, +1$.

a) Show that

$$Y_{1,-1}(\theta, \phi) := \sqrt{\frac{3}{8\pi}} \sin(\theta)\, e^{-i\phi}$$

satisfies

$$\hat{L}_{-} Y_{1,-1}(\theta, \phi) = 0.$$

b) Determine an expression for $Y_{1,0}(\theta, \phi)$.

c) Determine an expression for $Y_{1,1}(\theta, \phi)$.

d) Determine expressions for the angular probability densities for each and use these to describe the regions in which a particle is most likely to be located.

Answer: a)

$$\hbar e^{-i\phi}\left[-\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right] Y_{1-1}$$

$$= \hbar e^{-i\phi}\left[-\cos\theta\sqrt{\frac{3}{8\pi}}e^{-i\phi} + i\frac{\cos\theta}{\sin\theta}\sin\theta\sqrt{\frac{3}{8\pi}}e^{-i\phi}(-i)\right]$$

$$= \hbar e^{-i\phi}\sqrt{\frac{3}{8\pi}}e^{-i\phi}\left[-\cos\theta + \cos\theta\right] = 0$$

b)

$$L_+|1,-1\rangle = \hbar\sqrt{2-(-1)(0)}\,|1,0\rangle = \hbar\sqrt{2}\,|1,0\rangle$$

$$\Rightarrow Y_{10} = \frac{1}{\hbar\sqrt{2}}\hbar e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right]\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$$

$$= \frac{1}{\sqrt{2}}\, 1\, e^{i\phi}\sqrt{\frac{3}{8\pi}}\left[\cos\theta\, e^{-i\phi} + \cos\theta\, e^{-i\phi}\right]$$

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

c)

$$L_+|1,0\rangle = \hbar\sqrt{2}\,|1,1\rangle$$

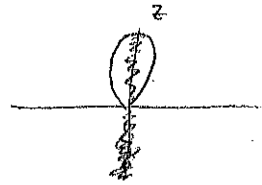
$$\Rightarrow Y_{11}(\theta,\phi) = \frac{-1}{\hbar\sqrt{2}}\hbar e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right]\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$= \frac{-1}{\sqrt{2}}e^{i\phi}\sqrt{\frac{3}{4\pi}}\sin\theta \quad \Rightarrow \quad Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

d) In all cases

$$P(\theta, \phi) = \sin\theta \, |Y_{\ell,m}(\theta,\phi)|^2$$

$\ell=1, m=-1$ $\quad P(\theta,\phi) = \sin^3\theta \, \frac{3}{8\pi}$

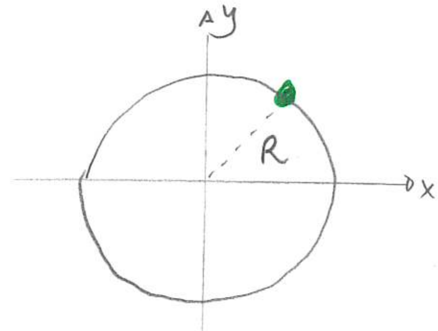



most likely along z axis

$\ell=1, m=0$ $\quad P(\theta,\phi) = \frac{3}{4\pi} \cos^2\theta \sin\theta$

max at intermediate angle...

$\ell=1, m=-1$ $\quad P(\theta,\phi) = \sin^3\theta \frac{3}{8\pi} \rightsquigarrow$ Most likely along z axis




## Application: Particle on a ring

Consider a quantum mechanical particle restricted to a ring in the x-y plane. Denote the mass of the particle by M and the radius of the ring by R. Then in a classical situation the energy is

$$E = \frac{1}{2} I \omega^2 = \frac{1}{2} M R^2 \omega^2$$

and in terms of the z-component of angular momentum:

$$E = \frac{1}{2I} L_z^2$$

where $I = MR^2$ is the moment of inertia. The comparable quantum system has Hamiltonian

$$\hat{H} = \frac{1}{2I} \hat{L}_z^2$$

This is a particle on a ring. Such nanostructures (often superconducting) can be created and populated with particles (e.g. electrons).

Bayer et.al PRL 90 186801 (2003)
Li PRB 83 115448 (2011)
Liu PRB 105 115426 (2022)

## 3 Quantum particle on a ring

A quantum particle on a ring has Hamiltonian

$$\hat{H} = \frac{1}{2I}\hat{L}_z^2$$

where $I$ has units of moment of inertia.

a) List the energy eigenstates of the system and the associated energies. Create an energy ladder diagram for the system.

b) Are the energies degenerate or not?

c) List the possible frequencies of electromagnetic radiation which could be emitted from such a system.

Answer: a) Consider states $|l,m\rangle$.

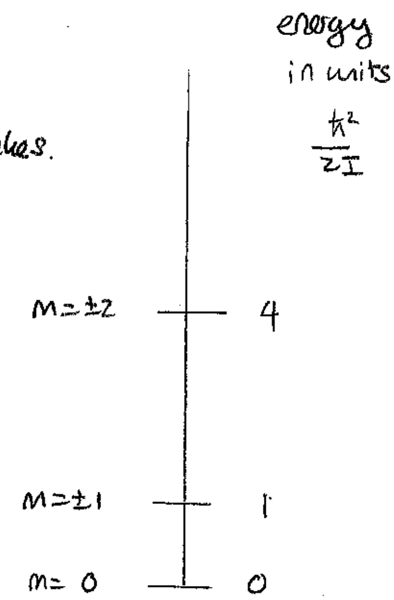
$$\hat{H}|l,m\rangle = \frac{1}{2I}\hat{L}_z^2|l,m\rangle$$

$$= \frac{1}{2I}\hat{L}_z \hbar m|l,m\rangle = \frac{\hbar m}{2I}\hbar m|l,m\rangle$$

$$\Rightarrow \hat{H}|l,m\rangle = \frac{\hbar^2}{2I}m^2|l,m\rangle,$$

This is an eigenstate with energy $\frac{\hbar^2}{2I}m^2$

In this case $\hat{L}^2 = \hat{L}_z^2$ and so the eigenvalues of $\hat{L}^2$ are constructed from $m^2$.




b) They are since $+m$ and $-m$ give the same energy.

c) We need a jump from $m_i$ to $m_f$

$$\Delta E = \frac{\hbar^2}{2I}(m_i^2 - m_f^2)$$

and possibilities for $m_i^2 - m_f^2$ are

1, 4, 9, 16 to $m_f = 0$

3, 8, 15 to $m_f = \pm 1$ $\Rightarrow$ 1, 3, 4, 5, 7, 8, 9, ...

5, 12, ... to $m_f = \pm 2$